Before doing more examples, let's review:

A stuff type is a functor

\[
\begin{array}{c}
X \\
F \\
F \downarrow \\
\text{FinSet}_0
\end{array}
\]

where \( X \) is a groupoid which we call the groupoid of "\( F \)-stuffed sets." The generating function of this stuff type is:

\[
|F|(z) = \sum_{n \in \mathbb{N}} |X_n| z^n
\]

where \( X_n \) is the groupoid of "\( F \)-stuffed \( n \)-element sets" so that:

\[
X = \sum_{n \in \mathbb{N}} X_n
\]

\[
\text{FinSet}_0 = \sum_{n \in \mathbb{N}} [\text{n-elt sets}]_0
\]

& we have

\[
\begin{array}{c}
X_n \\
F_n \\
\downarrow \\
[\text{n-elt sets}]_0
\end{array}
\]

\[
F_n = F|_{X_n}
\]
Our example so far was

\[ F = \text{"being a } C\text{-colored finite set"} \]

where \( C \) is a groupoid. Here

\[ X_n = \text{the groupoid of "} C\text{-colored } n\text{-elt sets"} \]

\[ F_n \quad \text{forgets the } C\text{-coloring} \]

\[ [n\text{-elt sets}]_0 \quad \text{---} \]

A morphism in \( X_n \) looks like:

\[
\begin{array}{ccc}
  c_1 & c_2 & c_3 \\
  \sigma_1 & \sigma_2 & \sigma_3 \\
  c'_1 & c'_2 & c'_3 \\
\end{array}
\]

\[ \sigma : \{1, 2, 3\} \to \{1, 2, 3\} \]

(or any 3-elt set)

\[ f_i : c_i \to c_{\sigma(i)} \]

are morphisms in \( C \)

So

\[ X_n \cong \frac{C^n}{n!} \]

\[ n! \text{ has an action on } C^n \text{ by "crossing the strands"} \]

objects are \( n\)-tuples of objects in \( C \); morphisms in \( X_n \) are \( n\)-tuples of morphisms in \( C \) composed with permutations

\[
\left( \begin{array}{ccc}
  f_1 & f_2 & f_3 \\
  \end{array} \right) = \left( \begin{array}{ccc}
  f_1 & f_2 & f_3 \\
  \end{array} \right)
\]
So:
\[ |F|(z) = \sum |X_n| z^n = \sum \left| \frac{c^n}{n!} \right| z^n = \sum \frac{|c|^n}{n!} z^n = e^{\mid c \mid z} \]

Q: What should this stuff type really be called?
A: "E\(^{CZ}\)" so that \(|E^{CZ}| = e^{\mid c \mid z}\).

Q: What does this really mean? i.e. if \(E^{CZ}\) is something like a "function," what sort of values does the argument \(Z\) take?
A: Groupoids! If \(Z_0\) is a groupoid, \(E^{CZ_0}\) should be a groupoid — the groupoid of "\(CZ_0\)-colored finite sets," i.e. finite sets with elts labelled by objects of \(CZ_0\), i.e. pairs \((z, z)\) with \(c \in C, z \in Z_0\).

Last quarter we saw that \(|E^{CZ_0}| = e^{\mid c \mid |Z_0|}\).

E.g.: \(C = \frac{1}{2!}\) (\(Z/2\) seen as a groupoid)

Then
\[ E^{CZ} = "\text{being a } \frac{1}{2} \text{-colored finite set}" \]
and if we evaluate this at \(Z_0 = 1\) (the 1 elt set, viewed as a groupoid) \(\Box\)
we get

\[ E^c = E^{1/2}! = \text{the groupoid of } \frac{1}{2} \text{-colored finite sets} \]

\[ \simeq \text{Cubes} \]

(see p. 67, Week 7, Winter 2004)

Back to our examples:

2) \( F = \text{"being the first of two finite sets of the same cardinality"} \)

We want to make this a stuff type

\[
\begin{array}{ccc}
X & \xrightarrow{F} & \text{FinSet}_0 \\
\end{array}
\]

where objects of \( X \) are pairs of sets \((S,T)\) with \( S \cong T \), and morphisms are pairs of bijections, and \( F \) forgets the second set and second morphism is these pairs. This is a stuff type but not a structure type because it's not faithful.

\[ X_n = \text{the groupoid of "pairs of n-elt sets"} \]

\[ \simeq [\text{n-elt sets}]_0 \times [\text{n-elt sets}]_0 \]

So

\[ |X_n| = |[\text{n-elt sets}]_0|^2 \]

\[ [\text{n-elt sets}] \simeq \frac{1}{n!} \quad \text{so} \quad |X_n| = \frac{1}{(n!)^2} \]

and so

\[ |F| = \sum_{n \in \mathbb{N}} \frac{z^n}{(n!)^2} \]

(which can be expressed using integrals of Bessel fns.)
Note: This couldn’t possibly be the generating function of any structure type, since if we write
\[ 1F(z) = \sum \frac{a_n}{n!} z^n \]
we don’t get \( a_n \in \mathbb{N} \)!
(Of course we are ignoring the detail that we haven’t shown our way of finding generating functions of stuff types is equivalent to our old way for structure types—we’ll deal with this later.)

Now we can fill in this chart:

| If \( F \) is a... | Then \( 1F(z) = \sum \frac{a_n}{n!} z^n \) where... | Since these numbers are cardinalities of:
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>stuff type</td>
<td>( a_n \in \mathbb{R}^+ = [0, \infty) )</td>
<td>(tame) groupoids = 1-groupoids</td>
</tr>
<tr>
<td>structure type</td>
<td>( a_n \in \mathbb{N} )</td>
<td>(finite) sets = 0-groupoids</td>
</tr>
<tr>
<td>property type</td>
<td>( a_n \in {0,1} \cong {F,\neg F} )</td>
<td>truth values = -1-groupoids</td>
</tr>
<tr>
<td>vacuous property type</td>
<td>( a_n \in {1} \cong {\neg F} )</td>
<td>true = the only -2-groupoid</td>
</tr>
</tbody>
</table>

The reason this all works in this way is that in equipping a finite set with extra \( \{ \text{stuff, structure properties, vacuous properties} \} \), there’s

\[ \{ \text{groupoid set, pair, 1-elt set} \} \], i.e. a \[ \{ 1\text{-groupoid, 0-groupoid, -1-groupoid, -2-groupoid} \} \] of choices.
Evaluation & Composition of Stuff Types

Following what we did last quarter for structure types, let's define $F(Z_0)$, where $F$ is a stuff type:

$$
X \rightarrow a \text{ groupoid}
$$

$F \downarrow$

$FinSet_0$

& $Z_0$ is any groupoid. Copying what we did, let

$$F(Z_0) = \text{ the groupoid of "F-stuffed } Z_0\text{-colored finite sets"}$$

just as before, but with "stuffed" instead of "structured"; now we're equipping the finite sets with extra stuff.

A typical object in $F(Z_0)$ looks like:

$\alpha$ (iso)morphism $\alpha : F(x) \rightarrow \{z_1, \ldots, z_n\}$

$x \in X$ is an object equipped with

- a label or "color" the elements of the finite set $\{z_1, \ldots, z_n\}$
- redundant since $FinSet_0$ is a groupoid

The "tag" is our cute depiction of the "F-stuff" $\alpha$ is the string that connects the "F-stuff" to the set.
A typical morphism in \( F(Z_0) \) looks like:

\[
\sigma: \{a_1, \ldots, a_n\} \rightarrow \{a_1', \ldots, a_n'\}
\]

\( f_i : z_i \rightarrow z_{i'} \) morphisms in \( Z_0 \)

\( g : x \rightarrow x' \) a morphism in \( X \)

such that

\[
\begin{array}{ccc}
F(x) & \xrightarrow{\alpha} & \{a_1, \ldots, a_n\} \\
F(g) & & F(x') \\
\downarrow & & \downarrow \sigma \\
F(x') & \xrightarrow{\alpha'} & \{a_1', \ldots, a_n'\}
\end{array}
\]

commutes

**Thm:** \(| F(Z_0) | = | F(\text{Id}_{Z_0}) | \)

**Proof:** Have you noticed we never prove theorems here? ■
Given two stuff types:

\[
\begin{array}{c}
X \\
\downarrow F \\
\text{FinSet}_o
\end{array}
\quad
\begin{array}{c}
Y \\
\downarrow G \\
\text{FinSet}_o
\end{array}
\]

can we compose them to get a stuff type \( F \circ G \) with

\[(F \circ G)(Z_o) \simeq F(G(Z_o))?
\]

Yes!

\[
F(G(Z_o)) = \text{the groupoid of "F-stuffed } G(Z_o) \text{-colored finite sets"}
\]

= \text{the groupoid of "F-stuffed finite sets with elements labelled by objects of } G(Z_o)\"

= \text{the groupoid of "F-stuffed finite sets with elements labelled by } G \text{-stuffed } Z_o \text{-colored finite sets."}
This groupoid is the same as:

\[(F \circ G)(Z_0) := \text{the groupoid of} \ "F \circ G\text{-stuffed } Z_0\text{-colored finite sets}"\]

provided we let

\[F \circ G = \text{"being a finite set } S \text{ written as a finite disjoint union } S_1 + \ldots + S_n\]
\[\text{with each } S_i \text{ equipped with } G\text{-stuff and } \{1,\ldots,n\} \text{ equipped with } F\text{-stuff,} \]
\[\text{corresponding to } S_1,\ldots,S_n\]
\[= \text{"being the disjoint union of a finite } F\text{-stuffed family of } G\text{-stuffed finite sets."}\]

**Examples:**

1) \(\text{COSH } Z\) = "being an even set"
\[E^Z = \text{"being a finite set"}\]
\[E^{\text{COSH } Z} = \text{"being a disjoint union of a finite family of even sets"}\]

\(\text{E.g.}\)
\[ E^{\text{COSH}}(2) = \text{the groupoid of "finite disjoint unions of } Z_0\text{-colored even sets"} \]

in particular:

\[ E^{\text{COSH}}(1) = \text{the groupoid of "finite disjoint unions of } 1\text{-colored even sets"} \]

\[ \cong \text{the groupoid of "finite disjoint unions of even sets"} \]

and

\[ |E^{\text{COSH}}(1)| = |E^{\text{COSH}}(1)| = e^{\text{COSH}(1)} = e^{\cosh 1} = e^{\frac{e+e^{-1}}{2}}. \]

What's \[ E^{Z_0\text{COSH}}(1) \]?

\[ \text{COSH}(1) \cong \text{the groupoid of "even sets"} \]

\[ Z_0\text{COSH}(1) \cong \text{the groupoid of pairs } (z, x) \text{ w/ } z \in Z_0 \text{ & } x \in \text{COSH}(1) \]

a typical example:

\[ z \in Z_0 \]

So \[ E^{Z_0\text{COSH}}(1) \cong \text{the groupoid of "finite sets w/ elts labelled by pairs } (z,x) \in Z_0\text{COSH}(1)". \]