Structure Types - New vs. Old Definition

How are our new structure types:

\[
\begin{align*}
X & \xrightarrow{\text{groupoid}} \\
\downarrow F & \xleftarrow{\text{faithful functor}} \\
\text{FinSet}_0 & 
\end{align*}
\]

Secretly the same as our old ones:

\[
\begin{array}{ccc}
\text{FinSet}_0 & \xrightarrow{\text{functor}} & \text{Set} \\
\end{array}
\]

We'll show how to get \( G = F^* \) from \( F \). In fact, we can even do something like this for arbitrary stuff types:

\[
\begin{align*}
X & \xleftarrow{\text{groupoid}} \\
\downarrow F & \xrightarrow{\text{functor}} \\
\text{FinSet}_0 & 
\end{align*}
\]

We want to define for any \( S \in \text{FinSet}_0 \) its fiber (i.e. its "inverse image" \( F^{-1}(S) \)) consisting of the set of all ways to put \( F \)-structure on \( S \), or more generally the groupoid of ways to put \( F \)-stuff on \( S \). If we do this, we'll get

\[
G = F^*: \text{FinSet}_0 \longrightarrow \text{Gpd}
\]

\[
\text{S} \longrightarrow F^{-1}(S)
\]
but when \( F \) is faithful this will reduce to:

\[
G = F^* : \text{FinSet}_0 \rightarrow \text{Set} \subseteq \text{Gpd}
\]

our original description of structure types.

So: given \( X \)
\[
\begin{array}{c}
\downarrow \quad F \\
\text{FinSet}
\end{array}
\]

& given \( S \in \text{FinSet}_0 \), let's define \( F^{-1}(S) \) to be the groupoid s.t.:

- an object of \( F^{-1}(S) \) is an object \( x \in X \) equipped with an isomorphism \( \alpha : F(x) \rightarrow S \)

- a morphism in \( F^{-1}(S) \) from \((x, \alpha)\) to \((x', \alpha')\) is a morphism \( f : x \rightarrow x' \) s.t.

\[
F(x) \xrightarrow{F(f)} F(x')
\]

commutes.

Technically we call \( F^{-1}(S) \) the weak inverse image of \( S \) under \( F \), or the homotopy fiber, following terminology from topology.
We can extend this process to get a weak 2-functor:

\[ F^*: \text{FinSet}_0 \longrightarrow \text{Gpd} \]

\[ S \mapsto F^{-1}(S) = F^*(S) \]

i.e. for any bijection \( f: S \rightarrow S' \) we get

\[ F^*(f): F^*(S) \rightarrow F^*(S') \]

s.t. \( F^*(ff') \) equals \( F^*(f)F^*(f') \) only up to specified natural isomorphism, which satisfy some coherence laws.

If \( F \) is a structure type, i.e. it's faithful, then \( F^{-1}(S) \) is (a groupoid that's equivalent to) a set. Then by tweaking \( F^* \) slightly to make \( F^{-1}(S) \) really be a set, we get

\[ F^*: \text{FinSet}_0 \longrightarrow \text{Set} \subseteq \text{Gpd} \]

— our old way of thinking about structure types.

Conversely, any "old" structure type gives us a "new" one. So...

A functor \( F: X \rightarrow \text{FinSet}_0 \) that forgets...

<table>
<thead>
<tr>
<th>stuff</th>
<th>[ F^*: \text{FinSet}_0 \rightarrow \text{Gpd} = 1\text{-Gpd} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>structure</td>
<td>[ F^*: \text{FinSet}_0 \rightarrow \text{Set} = 0\text{-Gpd} ]</td>
</tr>
<tr>
<td>properties</td>
<td>[ F^*: \text{FinSet}_0 \rightarrow {\emptyset, 1} = -1\text{-Gpd} ]</td>
</tr>
<tr>
<td>vacuous properties</td>
<td>[ F^*: \text{FinSet}_0 \rightarrow {1} = -2\text{-Gpd} ]</td>
</tr>
</tbody>
</table>
In short, there's a
\[
\begin{align*}
\text{groupoid} & \\
\text{set} & \\
\text{empty or 1-elt set} & \\
\text{1-elt set} & \end{align*}
\]
\text{of ways to equip a finite set with stuff structure property vacuous properly}
\text{of some type.}

\text{The Inner Product of Stuff Types}

First consider
\[
\ell^2 = \{ \psi : \mathbb{N} \to \mathbb{C} : \sum_{n \in \mathbb{N}} |\psi_n|^2 < \infty \}.
\]
This is a Hilbert space with inner product
\[
\langle \phi, \psi \rangle := \sum_{n \in \mathbb{N}} \overline{\phi_n} \psi_n
\]
Following our general strategy, we can replace \(\mathbb{C}\) by its combinatorial heart, \(\mathbb{N}\), getting:
\[
\{ \psi : \mathbb{N} \to \mathbb{N} : \sum_{n \in \mathbb{N}} \psi_n^2 < \infty \}
\]
with inner product
\[
\langle \phi, \psi \rangle := \sum_{n \in \mathbb{N}} \phi_n \psi_n \in \mathbb{N}
\]
Then we could categorify this a bit, getting:

\[ \exists \psi : \mathbb{N} \to \text{FinSet} : \left| \sum_{n \in \mathbb{N}} \psi_n \right| < \infty \]

Such a \( \psi \) looks like:

\[
\begin{array}{cccccc}
\psi_0 & \psi_1 & \psi_2 & \psi_3 & \psi_4 & \cdots \\
\circ & \circ & \bigcirc & \circ & \circ & \cdots
\end{array}
\]

\[ \text{only finitely many nonempty} \]

or:

\[
\begin{array}{c}
X \\
\downarrow \\
\mathbb{N}
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & \cdots
\end{array}
\]

where \( X \) is a finite set.

In fact \( \psi = F^* \) is defined in terms of \( F \) just as we'd seen before:

\[ \psi_n = F^{-1}(n) \]

- i.e. in terms of inverse images.

\[
F : X \to \mathbb{N} \quad \text{is "the same" as} \quad \psi : \mathbb{N} \to \text{FinSet} \quad \text{(s.t. } \left| \sum \psi_n \right| < \infty \text{)}
\]
This is just like:

\[ F : X \rightarrow \text{FinSet}_0 \] is "the same" as \[ \Psi : \text{FinSet}_0 \rightarrow \text{Gpd} \]

(c.s.t. something is finite...?)

We can use this analogy to define the inner product for stuff types.

The Inner Product of Stuff Types (cont.)

What we've seen is that any function \( F : E \rightarrow B \) can be seen as a "bundle" with "total space" \( E \) & "base space" \( B \):

\[
\begin{array}{c}
E \\
F \\
B
\end{array}
\]

with \( F \) the "projection." We can also think of it as a functor

\[ F^* : B \rightarrow \text{Set} \]

(viewing \( B \) as a category w/ only identity morphisms) assigning to each \( b \in B \) the "fiber" (inverse image)

\[ F^*(b) = F^{-1}(b) : \]

\[ \begin{array}{c}
\text{Set} \\
F^* \\
B
\end{array} \]

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The second way of thinking about functions seems more complicated, but that’s the one that comes up when you categorify $l^2$ a little bit, getting:

$$L^2 = \{ \Psi : \mathbb{N} \to \text{Set} : \sum |\Psi_n|^2 < \infty \}$$

Here $\Psi_n$ is the “set of ways for possibility $\# n$ to occur.” (Contrast this to $\Psi_n \in \mathcal{C}$ in ordinary QM — the “amplitude for possibility $\# n$ to occur.”) But let’s think about this “categorified $l^2$,” the first way: $\Psi = F^*$ for some bundle $F : X \to \mathbb{N}$

and the condition that $\Psi$ be normalizable: $\sum |\Psi|^2 < \infty$ becomes just the condition that $X$ is a finite set.
Suppose we have two elements \( \varphi, \psi \in L^2 \) corresponding to two "bundles" \( F : X \to \mathbb{N}, \ G : Y \to \mathbb{N} \) \((X, Y \in \text{FinSet})\):

\[
\varphi = F^* \quad \psi = G^*
\]

On the other hand, we can define their "inner product" to be:

\[
\langle \varphi, \psi \rangle = \sum_{n \in \mathbb{N}} \varphi_n \psi_n \in \text{FinSet}
\]

but how can we describe this in terms of \( F \& G \)?

\[
\varphi_n := F^{-1}(n)
\]
This is the "fiberwise product" of the bundles $F: X \to N$, $G: Y \to N$. The set $\langle \phi, \psi \rangle$ is called $X \times^\text{in} N Y$.

We have
\[
X \times^\text{in} N Y = \sum_{n \in \mathbb{N}} F^{-1}(n) \times G^{-1}(n) = \{ (x,y) \in X \times Y : F(x) = G(y) \} \]

and maps making the following diagram commute.

\[
\begin{array}{ccc}
X \times^\text{in} N Y & & \\
\downarrow P_x & & \downarrow P_y \\
X & & Y \\
\downarrow F & & \downarrow G \\
N & & \text{ } \\
\end{array}
\]

$P_x (x,y) = x$
$P_y (x,y) = y$

In fact, $X \times^\text{in} N Y$ is initial among such gadgets — it's called the pullback of

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow F & & \downarrow G \\
N & & \text{ } \\
\end{array}
\]
Let's recall the definition of pullback. In any category, a pullback of the diagram

\[
\begin{array}{ccc}
x & \rightarrow & y \\
\downarrow f & & \downarrow g \\
z & \rightarrow & & \\
\end{array}
\]

is an object \( a \) with morphisms \( p_x : a \rightarrow x \), \( p_y : a \rightarrow y \) s.t.

\[
\begin{array}{ccc}
a & \rightarrow & y \\
\downarrow p_x & & \downarrow p_y \\
x & \rightarrow & & \\
\end{array}
\]

\[
\begin{array}{ccc}
a' & \rightarrow & x \\
\downarrow & & \downarrow f \\
a & \rightarrow & & \\
\end{array}
\]

\[
\begin{array}{ccc}
a' & \rightarrow & y \\
\downarrow & & \downarrow g \\
a & \rightarrow & & \\
\end{array}
\]

commutes & \( a \) is initial among such: if \( a' \), \( p_x : a' \rightarrow x \), \( p_y : a' \rightarrow y \) is another one (making the analogous square commute) then \( \exists ! f : a \rightarrow a' \) s.t.

\[
\begin{array}{ccc}
a & \rightarrow & y \\
\downarrow p_x & & \downarrow p_y \\
x & \rightarrow & & \\
\end{array}
\]

\[
\begin{array}{ccc}
a' & \rightarrow & x \\
\downarrow & & \downarrow f \\
a & \rightarrow & & \\
\end{array}
\]

\[
\begin{array}{ccc}
a' & \rightarrow & y \\
\downarrow & & \downarrow g \\
a & \rightarrow & & \\
\end{array}
\]

commutes.

In the category \( \text{Set} \), we can take \( a = \{(x,y)\in x \times y : f(x) = g(y)\} \) — i.e. a fiberwise product!
Now that we know how to take the "inner product" of

$$F : X \rightarrow \text{IN} \quad \text{and} \quad G : Y \rightarrow \text{IN}$$

we can copy this and get the inner product of stuff types:

$$F : X \rightarrow \text{FinSet}_0 \quad \text{and} \quad G : Y \rightarrow \text{FinSet}_0$$

The inner product of stuff types $F$ and $G$ will be a groupoid $\langle F, G \rangle$. This will be defined as a weak pullback:

$$\langle F, G \rangle = X \times_{\text{FinSet}_0} Y$$

Now this will only commute up to the natural isomorphism $\alpha$, and only be weakly initial among such gadgets. It should come as no surprise that $X \times_{\text{FinSet}_0} Y$ can be defined as the groupoid where:
- an object is a pair \((x, y) \in X \times Y\) equipped with an isomorphism \(\alpha_{(x,y)} = \alpha : F(x) \simrightarrow G(y)\).

- a morphism is a morphism \((f, g) : (x,y) \rightarrow (x',y')\) in \(X \times Y\) such that:

\[
\begin{array}{ccc}
F(x) & \xrightarrow{\alpha} & G(y) \\
\downarrow F(f) & & \downarrow G(g) \quad \text{commutes.} \\
F(x') & \xrightarrow{\alpha'} & G(y')
\end{array}
\]