Wick Powers

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To get the combinatorics of Feynman diagrams to work nicely, it'll be best to use the **Wick** *or* **normal-ordered** *powers of the position operator instead of its ordinary powers. Roughly speaking, the Wick power* $:q^n:$ *is designed to be as much as possible like the ordinary power* q^n, except *for the fact that*

$$\langle 1, :q^n: 1\rangle = 0.$$

This will greatly simply calculations of transition amplitudes when we study perturbed versions of the harmonic oscillator — which counts for about 80% of what people actually do in quantum field theory. Your goal in this homework is to learn what these Wick powers are. But first, let me briefly remind you of everything you need to know to tackle this!

Recall that the **Weyl algebra**, **W**, *is the associative algebra over* $\mathbb{C}$ *generated by elements* q, p *satisfying this relation:*

$$pq - qp = -i,$$

or

$$[p, q] = -i$$

for short. The Weyl algebra has a number of interesting representations, but for now let us work in the Fock representation.

In the **Fock representation**, *we think of* p *and* q *as operators on the space* $\mathbb{C}[z]$ *consisting of complex polynomial in one variable* z. *To do this, we first define the* **annihilation operator** a *and* **creation operator** a^*, *as follows:*

$$(a\psi)(z) = \psi'(z)$$

$$(a^*\psi)(z) = z\psi(z).$$

These satisfy

$$[a, a^*] = 1.$$

Then we define the **position operator** q *and* **momentum operator** p *in terms of these:*

$$q = \frac{a + a^*}{\sqrt{2}}$$

$$p = \frac{a - a^*}{\sqrt{2}i}.$$

These satisfy $[p, q] = -i$, *as desired.*

In the homework k**-Colorings as Categorified Coherent States**, *we saw that* $\mathbb{C}[z]$ *has a unique inner product such that* $1 \in \mathbb{C}[z]$ *is a unit vector and*

$$\langle a^*\phi, \psi\rangle = \langle \phi, a\psi\rangle.$$

This inner product is given explicitly by

$$\langle z^n, z^m\rangle = n!\, \delta_{nm}.$$

The vector z^n has an important physical significance! When we normalize it, we get a state in which the quantum harmonic oscillator has n quanta of energy. More precisely, if we define the **harmonic oscillator Hamiltonian** *by*

$$H_0 = \frac{1}{2}(p^2 + q^2 - 1) = a^* a,$$

then we have

$$H_0 z^n = n z^n.$$

The state $z^0 = 1$ is especially important since it's the **ground state** *of the harmonic oscillator: that is, the state with the least energy.*

But enough reminders... on to something new!

It's easy to check that that the powers of q satisfy

$$[p, q^n] = -inq^{n-1}, \qquad [q, q^n] = 0.$$

The **Wick powers** *of q, denoted $:q^n:$, are elements of $\mathbf{W}$ satisfying*

$$[p, :q^n:] = -in{:}q^{n-1}{:}, \qquad [q, :q^n:] = 0$$

but also

$$\langle 1, :q^n: 1\rangle = 0.$$

This says that the expectation value of $:q^n:$ in the ground state of the harmonic oscillator is zero. By definition we have

$$:q^0: = 1,$$

but the higher Wick powers of q are, in general, different from the ordinary powers!

Your job is to figure out what they are.

1. Show that an element of the Weyl algebra is determined up to an additive constant by its commutators with p and q.

In other words, show that if w, w' are two elements of W with

$$[p, w] = [p, w'],$$

$$[q, w] = [q, w'],$$

then $w - w' = c1$ for some $c \in \mathbb{C}$.

Hint: I'll let you use without proof the fact that any element $w \in \mathbf{W}$ can be written as a 'polynomial'

$$w = \sum_{m,n \in \mathbb{N}} w_{mn} p^m q^n$$

for a unique choice of the coefficients $w_{mn} \in \mathbb{C}$. Use this to work out $[ip, w] = f$ and $[-iq, w] = g$, and check that f and g determine w up to an additive constant.

In doing this, you should see that the commutators $[ip, w]$ and $[-iq, w]$ are like the partial derivatives of w with respect to q and p, respectively. The above result is thus a quantum (i.e. noncommutative) version of the following fact: any polynomial $w \in \mathbb{C}[x, y]$ is uniquely determined up to an additive constant by its partial derivatives

$$\frac{\partial w}{\partial x} = f, \qquad \frac{\partial w}{\partial y} = g.$$

2. Using Problem 1, show that there is at most one sequence of elements $:q^n:\in \mathbf{W}$, the **Wick powers** of q, such that

$$[ip, :q^n:] = n:q^{n-1}:, \qquad [q, :q^n:] = 0,$$

$$\langle 1, :q^n: 1\rangle = 0$$

for $n > 0$, and

$$:q^0:= 1.$$

This proves the Wick powers are unique. *But why do they* exist? *For this, its easiest to give a nice* formula *for them...*

3. Show that if we define

$$:q^n:= (\frac{1}{\sqrt{2}})^n \sum_{k=0}^{n} \binom{n}{k} a^{*k} a^{n-k}$$

then

$$[ip, :q^n:] = n:q^{n-1}:, \qquad [q, :q^n:] = 0,$$

$$\langle 1, :q^n: 1\rangle = 0$$

for $n > 0$, and

$$:q^0:= 1.$$

By Problem 2, the Wick powers of q must be given by this formula.

This formula for $:q^n:$ should remind you of the binomial formula! We would get this formula if we used

$$q = \frac{a + a^*}{\sqrt{2}}$$

to expand q^n in terms of a and a^* and then brutally moved all the a's to the right of the a^*'s, as if they commuted! *This is why people also call $:q^n:$ the* **normal-ordered** *nth power of q:* **normal ordering** *means putting all the annihilation operators on the right.*

Of course the annihilation and creation operators don't *commute, so $:q^n:$ is not equal to q^n. But, they're close relatives. We make this more precise in the next exercise by showing that $:q^n:$ is a polynomial of degree n in q.*

4. Show inductively that there for each n there exists a polynomial P_n with

$$P_n'(q) = nP_{n-1}(q),$$

$$\langle 1, P_n(q)1\rangle = 0$$

for $n > 0$, and

$$P_0(q) = 1.$$

Show that $:q^n:= P_n(q)$.

Hint: once you know P_{n-1}, the equation $P_n' = nP_{n-1}$ determines everything about P_n except its constant term, which we can choose to obtain $\langle 1, P_n(q)1\rangle = 0$. Once you have polynomials P_n satisfying these equations, you can check using Problem 2 that $P_n(q) =:q^n:$. Remember that $[ip, \cdot]$ acts like the partial derivative $\frac{\partial}{\partial q}$.

It's still a bit of work to figure out explicitly what these polynomials P_n are. Up to some normalization factors, they turn out to be Hermite polynomials. Here are some examples:

$$
\begin{aligned}
:q^0: &= 1 \\
:q^1: &= q \\
:q^2: &= q^2 - \tfrac{1}{2} \\
:q^3: &= q^3 - \tfrac{3}{2}q \\
:q^4: &= q^4 - 3q^2 + \tfrac{3}{4}
\end{aligned}
$$

To see that these are correct, first note that the polynomials $P_n(q)$ on the right-hand side satisfy $P_n' = nP_{n-1}$. This determines each polynomial in terms of the previous one up to an additive constant. To check that we have the constant right we need to check $\langle 1, P_n(q)1\rangle = 0$ for $n > 0$. For this, we use this formula, which I'll prove in class:

$$
\langle 1, q^n 1\rangle = \begin{cases} 0 & n \text{ odd} \\ \frac{(n-1)!!}{(\sqrt{2})^n} & n \text{ even} \end{cases}
$$

for $n > 0$.

So, for example, to check the constant is right in the formula for $:q^4:$, we note

$$
\begin{aligned}
\langle 1, (q^4 - 3q^2 + \tfrac{3}{4})1\rangle &= \tfrac{3!!}{\sqrt{2}^4} - 3 \cdot \tfrac{1!!}{\sqrt{2}^2} + \tfrac{3}{4} \\
&= \tfrac{3}{4} - \tfrac{3}{2} + \tfrac{3}{4} \\
&= 0.
\end{aligned}
$$