$C^\infty$ has pullbacks.

**Definition of a smooth space**

A smooth space is a set $X$ equipped with, for each convex set $C$, a set of plots

$$\phi : C \to X$$

such that

- Given a plot $\phi : C \to X$ and a smooth map $f : C' \to C$ between convex sets, $\phi \circ f : C' \to X$ is a plot.
- Given inclusions $i_\alpha : C_\alpha \to C$ such that $\{C_\alpha\}$ is an open cover of $C$, given $\phi : C \to X$, then $\phi \circ i_\alpha : C_\alpha \to X$ are plots for every $\alpha$ implies $\phi : C \to X$ is a plot.
- Every map from a point (in $\mathbb{R}^n$) to $X$ is a plot.

**The pullback**

Let $X, Y, Z$ be smooth spaces with smooth maps

$$f : X \to Z \quad \text{and} \quad g : Y \to Z.$$ 

Now consider the product $X \times Y$, a smooth space, where a function

$$\phi : C \to X \times Y$$

is a plot if and only if

$$\begin{align*}
C &\xrightarrow{\phi} X \times Y \xrightarrow{p_1} X \\
C &\xrightarrow{\phi} X \times Y \xrightarrow{p_2} Y
\end{align*}$$

are plots.

We will show that the pullback of the diagram
is the subset of $X \times Y$

$$P = \{ z \in X \times Y \mid fp_1(z) = gp_2(z) \},$$

where a plot $\phi: C \to P$ is any function such that

$$C \xrightarrow{\phi} P \hookrightarrow X \times Y$$

is a plot.

**The universal property**

Suppose $Q$, a smooth space, is a competitor for the title of ‘the’ pullback. Then it must come equipped with smooth maps $q_1, q_2$ such that

$$\begin{array}{ccc}
Q & \xrightarrow{q_1} & X \\
\downarrow{q_2} & \searrow{f} & \\
Y & \xrightarrow{g} & Z
\end{array}$$

commutes. Consider the map $\phi: Q \to P$ defined by $r \mapsto (q_1(r), q_2(r))$. By construction this is the unique map such that the following diagram commutes.

$$\begin{array}{ccc}
Q & \xrightarrow{\phi} & P \\
\downarrow{q_2} & \searrow{p_1} & \swarrow{p_2} & \nearrow{f} \\
Y & \xrightarrow{g} & X & \xrightarrow{f} & Z
\end{array}$$

So we just need to show that $\phi$ is a morphism in our category, i.e. that it is smooth. Recalling the definition, $\phi$ is smooth if for every plot $\alpha: C \to Q$, $\phi \circ \alpha: C \to P$ is a plot. Since $P$ is a subset of the product space $X \times Y$, $\phi \circ \alpha$ is a plot if and only if

$$\begin{array}{ccc}
C & \xrightarrow{\alpha} & Q & \xrightarrow{\phi} & P & \xrightarrow{p_1} & X \\
C & \xrightarrow{\alpha} & Q & \xrightarrow{\phi} & P & \xrightarrow{p_2} & Y
\end{array}$$

are plots. This follows from the fact that

$$\begin{array}{ccc}
C & \xrightarrow{\alpha} & Q & \xrightarrow{q_1} & X \\
C & \xrightarrow{\alpha} & Q & \xrightarrow{q_2} & Y
\end{array}$$

are plots and that
commutes.