COURSE NOTES ON QUANTIZATION AND COHOMOLOGY, SPRING 2007

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1. Preface

These are lecture notes taken at U.C. Riverside, in the Tuesday lectures of John Baez's Quantum Gravity Seminar, Spring 2007. Apoorva Khare typed notes and Christine Dantas prepared figures based on handwritten notes by Derek Wise. This is the third and final quarter of a year-long course. You can find the latest version of notes for all three quarters here:

`http://math.ucr.edu/home/baez/qg-spring2007/`

If you see typos or other problems with any of these notes, please let John Baez know (`baez@math.ucr.edu`).

2. Apr 3, 2007: Smooth categories

We've been starting with a category $\mathcal{C}$ equipped with an "action" functor

$$S : \mathcal{C} \to (\mathbb{R}, +)$$

and doing classical and quantum mechanics using this.

To do quantum mechanics, we formed

$$e^{iS} : \mathcal{C} \to (U(1), \cdot)$$

and did path integrals - integrals over objects and/or morphisms of $\mathcal{C}$ - so we needed something like a measure space (or perhaps a generalized measure space) of objects and a (generalized) measure space of morphisms.

In classical mechanics, instead of integrating, we *minimize* the action, or find critical points of the action

Remark 2.1. Note that minimization requires *no* extra structure on $\mathcal{C}$, whereas finding critical points requires something like a "smooth structure" on the sets $\mathrm{Ob}(\mathcal{C})$ of objects and (especially) $\mathrm{Mor}(\mathcal{C})$ of morphisms. For example, we might want $\mathrm{Ob}(\mathcal{C}), \mathrm{Mor}(\mathcal{C})$ or each $\mathrm{Hom}_{\mathcal{C}}(x, y) \subset \mathrm{Mor}(\mathcal{C})$ to be a smooth manifold. If $\mathrm{Hom}_{\mathcal{C}}(x, y)$ were a manifold, we could demand that $S : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathbb{R}$ be smooth, and look for $\gamma \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ with

$$dS(\gamma) = 0$$

Alas, in examples, $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is usually a more general "smooth space".

2.1. **Example: Path groupoids.**

The path groupoid PM of a manifold M has

- points of M as objects: $\mathrm{Ob}(PM) = M$.
- thin homotopy classes of smooth paths $\gamma : [0, 1] \to M$ which are constant near 0 and 1, as morphisms.

Here, a *thin homotopy* between paths $\gamma_0, \gamma_1 : [0, 1] \to M$ is a smooth map:

$$H : [0, 1] \times [0, 1] \to M$$

such that

(1) $H(t, 0) \equiv \gamma_0(t)$.
(2) $H(t, 1) \equiv \gamma_1(t)$. (These two together explain the homotopy.)
(3) $H(0, s)$ is independent of s.
(4) $H(1, s)$ is independent of s.

and

(5) H is *thin*, i.e. $\mathrm{rank}(dH) \leq 1$ (i.e. the homotopy "sweeps out no area"), or equivalently, $\det(dH) = 0\ \forall (t, s)$.

Thin homotopies can do this:

[figure: point isomorphically maps to two points connected by an edge "traveling" both ways]

Also, any reparametrization is a thin homotopy, so the obvious composition of morphisms in PM is associative and has identities and inverses (trace "backwards" along any path, staying near the original point for some time at the beginning and at the end, to get the "thin" inverse path, or a composite path that is thin homotopic to the identity map - one that stays at the original point throughout.)

Thus, PM is a groupoid.

If α is a 1-form on M, we get a functor

$$S : PM \to (\mathbb{R}, +)$$

sending every object of PM to the one object of $(\mathbb{R}, +)$, and every morphism $[\gamma]$ to

$$S([\gamma]) := \int_\gamma \alpha \in \mathbb{R}$$

Note that if γ_0, γ_1 are thinly homotopic, then

$$\int_{\gamma_0} \alpha - \int_{\gamma_1} \alpha = \int_H d\alpha = 0$$

since $\det(dH) = 0$. Thus S is well-defined.

2.2. **Introduction to smooth categories.** This kind of example arises all over in classical mechanics, especially when M is a cotangent bundle and α is the canonical 1-form.

We would like to say that PM is a "smooth category", $S : PM \to (\mathbb{R}, +)$ is a "smooth functor", and $\mathrm{Hom}_{PM}(x, y) \subset \mathrm{Mor}(PM)$ is a "smooth space", so that we can find critical points.

Suppose we knew what "smooth spaces" and "smooth maps between smooth spaces" were. They'd better form a category, $\mathcal{C}^\infty$, say. Then, what's a "smooth category" and a "smooth functor"?

A smooth category $\mathcal{C}$ should have

(1) a smooth space of objects: $\mathrm{Ob}(\mathcal{C}) \in \mathcal{C}^\infty$.
(2) a smooth space of morphisms: $\mathrm{Mor}(\mathcal{C}) \in \mathcal{C}^\infty$.
(3) a smooth identity-assigning map $i : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$, i.e. $i$is a morphism in $\mathcal{C}^\infty$.
(4) smooth source and target maps $s, t : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$.
(5) a smooth "composition" map:

$$\circ : \mathrm{Mor}(\mathcal{C})_t \times_s \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$$

where $\mathrm{Mor}(\mathcal{C})_t \times_s \mathrm{Mor}(\mathcal{C})$ is the smooth space of composable pairs of morphisms in $\mathcal{C}$ (i.e. $\{(f, g) : t(f) = s(g)\}$ - so that $g \circ f = \xrightarrow{f} \xrightarrow{g}$

is defined), i.e. the pullback of the diagram

$$\begin{tikzcd} & \mathrm{Mor}(\mathcal{C}) \arrow[d, "s"'] \\ \mathrm{Mor}(\mathcal{C}) \arrow[r, "t"] & \mathrm{Ob}(\mathcal{C}) \end{tikzcd}$$

i.e. the universal object making

$$\begin{tikzcd} \mathrm{Mor}(\mathcal{C})_t \times_s \mathrm{Mor}(\mathcal{C}) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & \mathrm{Mor}(\mathcal{C}) \arrow[d, "s"'] \\ \mathrm{Mor}(\mathcal{C}) \arrow[r, "t"] & \mathrm{Ob}(\mathcal{C}) \end{tikzcd}$$

commute. Here, π_1, π_2 denote the projection to the first/second component respectively. (So $\mathcal{C}^\infty$ should have pullbacks or at least *this* pullback. The category of smooth manifolds and smooth maps doesn't have pullbacks!)

(6) the associative law, written as a commutative diagram in $\mathcal{C}^\infty$. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Mor}(\mathcal{C})_t \times_s \mathrm{Mor}(\mathcal{C})_t \times_s \mathrm{Mor}(\mathcal{C}) \arrow[r, "1\times\circ"] \arrow[d, "\circ\times 1"'] & \mathrm{Mor}(\mathcal{C})_t \times_s \mathrm{Mor}(\mathcal{C}) \arrow[d, "\circ"'] \\ \mathrm{Mor}(\mathcal{C})_t \times_s \mathrm{Mor}(\mathcal{C}) \arrow[r, "\circ"] & \mathrm{Mor}(\mathcal{C}) \end{tikzcd}$$

(7) the left and right unit laws.

(8) the source and target of a composite morphism, to be what they *should* be.

3. Apr 10, 2007: Smooth spaces

Last time we posed a question: what's a "smooth functor" $S : \mathcal{C} \to \mathbb{R}$? For this, we need $\mathcal{C}$ to be a "smooth category", which we succeeded in defining, *given* a category $\mathcal{C}^\infty$ of "smooth spaces" and "smooth maps". More generally, given a category K, we can define a *category in* K (say $\mathcal{C}$) to consist of

- $\mathrm{Ob}(\mathcal{C}) \in K$
- $\mathrm{Mor}(\mathcal{C}) \in K$
- $s, t : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ in K, i.e. $s, t \in \hom_K(\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$
- $i : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ in K
- $\circ : \mathrm{Mor}(\mathcal{C})_t \times_s \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ in K

satisfying the usual category theory axioms.

This trick - taking a definition and replacing sets and functions with objects and morphisms of K - is called *internalization* (in K). The result is often called an *internal category* (in K) or simply a *category in* K.

This works best if K has pullbacks, so we can write "$\mathrm{Mor}(\mathcal{C})_t \times_s \mathrm{Mor}(\mathcal{C})$" and know it's defined.

(We now look at examples.) The category $K = \mathtt{Set}$ has pullbacks:

$$\begin{tikzcd} X_f \times_g Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

where in this case,

$$X_f \times_g Y := \{(x, y) \in X \times Y : f(x) = g(y)\}$$

The category $\mathtt{Diff}$ of smooth manifolds and smooth maps doesn't have pullbacks. For a rough idea why, consider

$$\begin{tikzcd} \{(x, y) : f(x) = g(y)\} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X = \mathbb{R} \arrow[d, "f"] \\ Y = \mathbb{R} \arrow[r, "g"] & Z \end{tikzcd}$$

where $g(x) \equiv 0$, and

$$f(x) = \begin{cases} e^{-1/x}, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

[figure: the graph of $f(x)$]

Here, $\{(x, y) : f(x) = g(y)\} = \{(x, y) : x \leq 0\}$ is a manifold with boundary - *not* a smooth manifold.

A more sophisticated example. $f(x) = 0$ only on the Cantor set, $g(x) \equiv 0$.

[figure: graph of such an f]

So, let's introduce a bigger category of smooth spaces that *does* have pullbacks, following Grothendieck's dictum: *a nice category with some bad objects is better than a bad category with only nice objects.* He used this to invent "schemes", generalizing "algebraic varieties", and revolutionizing algebraic geometry.

Let's do the same for differential geometry. So, what's a smooth space? Following Chen's ideas:

Definition 3.1.

(1) A *convex set* is a convex subset of $\mathbb{R}^n$. For example,

[mini-figures: $\mathbb{R}^n$, a half-space, a quarter-space, etc.]

(Note: Convex sets need not be open.)

(2) A function $f : C \to C'$ between convex sets is *smooth* if it has continuous nth derivatives for all $n \geq 0$, defined in the usual way.

(3) A *smooth space* is a set X equipped with, for each convex set C, a set of *plots*

$$\varphi : C \to X$$

(which we think of as "smooth"), so that

  (a) Given a plot $\varphi : C \to X$ and a smooth map $f : C' \to C$ between convex sets, $\varphi \circ f : C' \to X$ is a plot.

  (b) Suppose we are given inclusions $i_\alpha : C_\alpha \to C$ such that $\{C_\alpha\}$ is an open cover of C, and a (set) map $\varphi : C \to X$. If $\varphi \circ i_\alpha : C_\alpha \to X$ is a plot for each α, then $\varphi : C \to X$ is a plot.

  (c) Every map from a point (in $\mathbb{R}^n$) to X is a plot.

(4) Given smooth spaces X, Y, a function $f : X \to Y$ is a *smooth map* if, for every plot $\varphi : C \to X$, the composite $f \circ \varphi : C \to Y$ is a plot in Y.

(5) The category of smooth spaces and smooth maps above, is denoted by $\mathcal{C}^\infty$. (Note that we need to verify that smooth maps compose to give smooth maps - but this is easy.)

Examples:

(1) Any convex set C becomes a smooth space, where the plots $\varphi : C' \to C$ are just the smooth maps (as defined earlier).
(2) Any set X has a *discrete* smooth structure such that the plots $\varphi : C \to X$ are just the constant functions.
(3) Any set X has an *indiscrete* smooth structure where every function $\varphi : C \to X$ is a plot.
(4) Any smooth manifold X becomes a smooth space where $\varphi : C \to X$ is a plot if and only if φ is smooth in the usual sense of smooth manifolds. Moreover, if X, Y are manifolds, $f : X \to Y$ is smooth according to our new definition if and only if it's smooth in the usual sense.

We now prove some functoriality results in $\mathcal{C}^\infty$.

Theorem 3.2. *$\mathcal{C}^\infty$ has products and coproducts, and is closed under taking subsets.*

Proof. The product $X \times Y$ of smooth spaces becomes a smooth space where a function $\varphi : C \to X \times Y$ is a plot if and only if the composites $C \xrightarrow{\varphi} X \times Y \xrightarrow{p_1} X$ and $C \xrightarrow{\varphi} X \times Y \xrightarrow{p_2} Y$ are plots. Thus $\mathcal{C}^\infty$ has products (note that the maps $p_1, p_2 : X \times Y \to X, Y$ respectively, are indeed smooth maps, and hence in $\mathcal{C}^\infty$).

Similarly, the disjoint union $X + Y := X \coprod Y$ of smooth spaces, becomes a smooth space where a function $\varphi : C \to X + Y$ is a plot iff either

$$[\text{commutative diagram}: C \xrightarrow{\varphi_1} X \xrightarrow{i_1} X + Y \text{ equals } \varphi : C \to X + Y]$$

commutes for some plot $\varphi_1 : C \to X$, or

$$[\text{commutative diagram}: C \xrightarrow{\varphi_2} Y \xrightarrow{i_2} X + Y \text{ equals } \varphi : C \to X + Y]$$

commutes for some plot $\varphi_2 : C \to Y$. Thus $\mathcal{C}^\infty$ has coproducts. (Once again, the maps : $X, Y \to X + Y$ are smooth by definition.)

Finally, any subset $Y \subset X$ of a smooth space X becomes a smooth space (and the inclusion a smooth map), where a plot $\varphi : C \to Y$ is any function such that

$$C \xrightarrow{\varphi} Y \hookrightarrow X$$

is a plot in X. (So for example, the Cantor set is a smooth space.) ∎

Homework. Show the following result.

Theorem 3.3. *$\mathcal{C}^\infty$ has pullbacks.*

Proof. Given a pair of morphisms in $\mathcal{C}^\infty$

$$\begin{tikzcd} & X \arrow[d, "f"'] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

define $X_f \times_g Y$ to be

$$X_f \times_g Y := \{(x,y) \in X \times Y : f(x) = g(y)\} \subset X \times Y$$

By the above theorem, this is a smooth space. We now show that $X_f \times_g Y \xrightarrow{p_1} X$ is a smooth map (the projection to Y is also smooth for similar reasons). Clearly, this map is a composite of the inclusion map : $X_f \times_g Y \hookrightarrow X \times Y$ and $p_1 : X \times Y \to X$ - but both of these maps are smooth (as shown above).

To complete the proof, we show that $X_f \times_g Y$ satisfies the universal property of pullbacks: given $W \in \mathcal{C}^\infty$ so that the diagram

$$\begin{tikzcd} W \arrow[r, "r_X"] \arrow[d, "r_Y"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

commutes in $\mathcal{C}^\infty$, we form $W \xrightarrow{r_X \oplus r_Y} X \oplus Y = X \times Y$. It can be seen that $r_X, r_Y, r_X \oplus r_Y$ are all smooth maps, and (set-theoretically, for instance, that) $\mathrm{im}(r_X \oplus r_Y) \subset X_f \times_g Y$, so that the obvious diagram for universality, now commutes. $\square$

4. Apr 17, 2007: The path groupoid of a smooth space

We've defined a notion of "smooth space" - a set X equipped with a collection of plots $\varphi : C \to X$ for convex sets C, satisfying some properties. We've also defined a notion of "smooth map" - a function $f : X \to Y$ (with X, Y smooth spaces) such that

$$\varphi : C \to X \text{ is a plot}$$

implies that

$$f \circ \varphi : C \to Y \text{ is a plot.}$$

We claimed that there's a category of smooth spaces and smooth maps. We listed a bunch of nice properties of $\mathcal{C}^\infty$; here are some more.

(1) (This is dual to the notion of subsets.) Any *quotient* $Y = X/\sim$, where X is a smooth space and $\sim$ is an equivalence relation on X, becomes a smooth space, where a plot in Y is a composite

$$C \xrightarrow{\varphi} X \xrightarrow{p} Y = X/\sim$$

where φ is any plot in X.

- This definition guarantees that the quotient map $p : X \to Y$ is smooth.
- This definition also lets us see that $\mathcal{C}^\infty$ has *pushouts*:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_1"] \\ Y \arrow[r, "i_2"] & X \coprod_\circ Y \end{tikzcd}$$

where the *pushout* $X \coprod_\circ Y$ is defined to be $(X \coprod Y)/\{i_1(f(z)) = i_2(g(z)) : z \in Z\}$.
The pushout is a quotient of a disjoint union $X + Y = X \coprod Y$ (i.e. the coproduct).

(2) $\mathcal{C}^\infty$ is *Cartesian-closed*: that is, given smooth spaces X and Y, the set

$$\mathcal{C}^\infty(X, Y) := \{f : X \to Y : f \text{ is smooth }\}$$

can be made into a smooth space such that there's a natural isomorphism

$$\mathcal{C}^\infty(X \times Y, Z) \cong \mathcal{C}^\infty(X, \mathcal{C}^\infty(Y, Z))$$

under $f \mapsto \widetilde{f}$, where in this case,

$$\widetilde{f}(x)(y) = f(x, y).$$

To do this, we define a plot in $\mathcal{C}^\infty(Y, Z)$, say

$$\varphi : C \to \mathcal{C}^\infty(Y, Z)$$

to be any map of the form $\widetilde{f}$, where

$$f : C \times Y \to Z$$

is smooth.

(Note that one must check that $\mathcal{C}^\infty(Y,Z)$ really is a smooth space and that then we get a one-to-one and onto map

$$\mathcal{C}^\infty(X\times Y,Z)\cong\mathcal{C}^\infty(X,\mathcal{C}^\infty(Y,Z))$$

which turns out to be smooth.)

Using these properties above, we can make a smooth category PX, the *path groupoid* of any smooth space X. This has

- a smooth space of objects $\mathrm{Ob}(PX)=X$.
- a smooth space of morphisms $\mathrm{Mor}(PX)=\mathcal{C}^\infty([0,1],X)/\sim$, where $[0,1]$ is smooth since it's a convex set, $\mathcal{C}^\infty([0,1],X)$ is smooth by Cartesian closedness, and modding out by "thin homotopy" gives a smooth space $\mathcal{C}^\infty([0,1],X)/\sim$. We also need to check that the source, target, identity-assigning, and composition maps are smooth.

Next, we want to understand smooth functors

$$S:PX\to(\mathbb{R},+)$$

since such a functor describes the "action" for some classical system.

Definition 4.1. Given smooth categories $\mathcal{C}$ and $\mathcal{D}$, a *smooth functor* is

- a smooth map $F:\mathrm{Ob}(\mathcal{C})\to\mathrm{Ob}(\mathcal{D})$
- a smooth map $F:\mathrm{Mor}(\mathcal{C})\to\mathrm{Mor}(\mathcal{D})$

such that the usual properties of a functor hold, written as commutative diagrams:

(1) F preserves the source for morphisms:

$$\begin{array}{ccc}\mathrm{Mor}(\mathcal{C}) & \xrightarrow{F} & \mathrm{Mor}(\mathcal{D})\\ {\scriptstyle s_C}\big\downarrow & & {\scriptstyle s_D}\big\downarrow\\ \mathrm{Ob}(\mathcal{C}) & \xrightarrow{F} & \mathrm{Ob}(\mathcal{D})\end{array}$$

and similarly,

(2) F preserves targets.
(3) F preserves the identity-assigning map.
(4) F preserves composition.

(Just as you can define "categories in K" for any category K with pullbacks, you can define "functors in K".)

Theorem 4.2. *For any smooth space M, there's a one-to-one correspondence between smooth functors*

$$S:PM\to(\mathbb{R},+)$$

and 1-forms A on M (defined below), given by

$$S([\gamma])=\int_\gamma A.$$

where we have

Definition 4.3. A *p-form* α on a smooth space X assigns to each plot $\varphi : C \to X$ a p-form α_φ on C, such that given a smooth map $f : C' \to C$ between convex sets,

$$f^*\alpha_\varphi = \alpha_{\varphi\circ f}$$

where f^* is the operation of pulling back p-forms from C to C' along $f : C' \to C$.

(For a proof of the theorem, see *Higher Gauge Theory - II* by Baez and Schreiber.)

5. May 1, 2007: Smooth functors and beyond

We've seen that if M is a smooth space (e.g. a manifold), then there is a smooth category PM, where

- objects are points $x \in M$;
- morphisms $\gamma : x \to y$ are thin homotopy classes of smooth maps $f : [0,1] \to M$ with $f(0) = x, f(1) = y$, and f constant near $0, 1$.

We think of PM as a category of "configurations" and "processes" for some physical system. So to formulate the Lagrangian approach to the physics of this system, we need a smooth functor

$$S : PM \to \mathbb{R}$$

describing the "action" of any process. Last time, we saw a Theorem that gave an explicit one-to-one correspondence between smooth functors $S : PM \to \mathbb{R}$ and 1-forms A on M, via

$$S([\gamma]) = \int_\gamma A$$

where we pick up a representative (path) for γ to define the integral.

Alas, this isn't general enough... as we'll soon see.

5.1. Quantum physics and circle-bundles.

To do quantum physics, what matters is not

$$S : PM \to \mathbb{R}$$

but the phase

$$e^{iS} : PM \to U(1)$$

which has less information, since $\exp : \mathbb{R} \to U(1)$ is many-to-one. In fact, e^{iS} is also sufficient to do classical physics!

[figure: path from x to y with "perturbations" in the middle, and accompanied by the equation: $\frac{d}{ds} S(\gamma_s) = 0$]

If we seek *critical points* of the action (instead of minima), we can work with e^{iS} instead of S:

$$\frac{d}{ds} e^{iS(\gamma_s)} = 0$$

(for all smooth homotopies γ_s of γ, that hold the endpoints fixed). This is because the critical points of e^{iS} are the same as those of S.

So this doesn't seem like a big deal.

Theorem 5.1. *There's a one-to-one correspondence between smooth functors*

$$P : PM \to U(1)$$

and 1-forms A on M, given by

$$P([\gamma]) = e^{i\int_\gamma A}$$

Here, P stands for "phase". So far the picture looks like:

[figure: cylinder along $x \to y$ on the X-axis, which plots points of M, and there's a path from one circular face of the cylinder (at x) to the other circular face (at y)]

For each point $x \in M$, we have a circle of possible phases for the system in configuration x, so we have a "trivial principal $U(1)$-bundle":

$$\begin{array}{ccc} M \times U(1) & \ni & (x,\alpha) \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_1} \\ M & \ni & x \end{array}$$

Sitting over $x \in M$, we have a *fiber*

$$\pi_1^{-1}(x) \subset M \times U(1)$$

which is a circle - the set of possible phases our system could have at x.

This example is called "trivial" because each fiber *is* $U(1)$ - or is canonically isomorphic to $U(1)$:

$$\pi_1^{-1}(x) = \{(x,\alpha) : \alpha \in U(1)\} \xrightarrow{\sim} U(1)$$

where the isomorphism sends (x,α) to α.

More interesting are the *nontivial* principal $U(1)$-bundles:

[figure: sphere, $M = S^2 = \mathbb{C}P^1$, with fiber described below]

For example, let the fiber over x be the set of points in S^2 that are perpendicular to x. Then we can't smoothly identify all the fibers with $U(1)$, since that would produce a nowhere vanishing smooth vector field on S^2!

[figure: the figure of a possible(?!) sphere with such a vector field]

5.2. **Torsors and principal bundles.** Now let's get a bit more formal. What's the difference between *a* circle and *the* circle? The circle is $U(1) \subset \mathbb{C}$. *A* circle is a "$U(1)$-torsor" - a copy of $U(1)$ that's forgotten what the element 1 is. (We now recall lots of stuff from the end of last quarter.)

Definition 5.2. For any group G, a *G-torsor* is a set X equipped with an action (a right action) of G:

$$\begin{array}{rcl} \alpha : X \times G & \to & X \\ (x,g) & \mapsto & xg \end{array}$$

so that $x1 = x$, and $(xg)h = x(gh)$, and such that X is isomorphic to G as a space with a right G-action.

In other words, there's a bijection

$$\beta : X \to G$$

so that

$$\beta(xg) = \beta(x)g$$

If $G = U(1)$, the difference between right and left action is inessential, since $U(1)$ is abelian. More importantly, any circle equipped with the ability to rotate it by any phase $g \in U(1)$, is a $U(1)$-torsor.

[figure: circle with a fixed point x, and a varying point y on the circle orthogonal to it; so $P \xrightarrow{\pi} M$ with $(x, y) \mapsto x$]

If a point in P is a point $x \in S^2$ together with a point in the circle perpendicular to x, and $\pi : P \to M$ is the obvious map, then $\pi^{-1}(x)$ is a $U(1)$-torsor.

More precisely, $\pi^{-1}(x)$ *becomes* a $U(1)$-torsor after we pick a "right- or left- hand rule" for rotating $y \in \pi^{-1}(x)$ by a phase $g \in U(1)$.

Getting back to principal $U(1)$-bundles, they are (among other things) smooth spaces P with a smooth map

$$\begin{array}{c} P \\ {\scriptstyle\pi}\big\downarrow \\ M \end{array}$$

such that each fiber $\pi^{-1}(x)$ (where $x \in M$) is equipped with the structure of being a $U(1)$-torsor.

Next time we'll *really* define a "principal $U(1)$-bundle", and include a clause saying that the $U(1)$-torsor structure on $\pi^{-1}(x)$ varies smoothly with x.

6. May 8, 2007: Smooth functors vs. connections on principal bundles

We're generalizing the idea of a smooth functor

$$e^{iS} : PM \to U(1)$$

to the concept of "parallel concept in a principal $U(1)$-bundle".

6.1. **Principal bundles - the formal definition.**

We first define principal $U(1)$-bundles, as promised last time; we give the definition in greater generality.

Definition 6.1. For any Lie group G, a *principal G-bundle over the smooth space M* is a smooth space P equipped with a (smooth) right action of G

$$\alpha : G \times P \to P$$

and a map

$$\begin{array}{ll} P & \text{[figure: tubular cylinder and circular cross section]} \\ \pi \big\downarrow & \\ M & \text{[figure: the base space } M \text{ and a point } x \text{ in it]} \end{array}$$

such that

(1) the action α preserves the *fibers* $\pi^{-1}(x)$ $\forall x \in M$ (i.e. for all $p \in \pi^{-1}(x)$, $g \in G$, we have $\alpha(p, g) \in \pi^{-1}(x)$); and
(2) $\pi : P \to M$ is *locally trivial*: $\forall x \in M$, there exists open $U \ni x$ and an isomorphism of right G-spaces $\gamma : \pi^{-1}(U) \to U \times G$ such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\gamma"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

commutes.

Remark 6.2. This implies that every fiber $P_x := \pi^{-1}(x)$ is a right G-space, but in fact we have an isomorphism of right G-spaces

$$\gamma : P_x \to \{x\} \times G \cong G$$

so P_x is a G-torsor!

6.2. **Parallel transport and connections.** Now let's talk about parallel transport - a new way. Given a principal G-bundle $\pi : P \to M$, we can form the *transport groupoid* $\mathrm{Trans}(P)$, where:

- objects are points $x \in M$ (or if you prefer, fibers P_x);
- morphisms $f : x \to y$ are maps (hence isomorphisms - since a G-set map between G-torsors is an isomorphism) between the torsors P_x and P_y.

Example. When $G = U(1)$, we think of P as a "circle bundle" over M, giving one circle of possible phases P_x for each $x \in M$. Then $\mathrm{Trans}(P)$ has these circles as objects, and "rotations" from one circle to another, as morphisms.

Definition 6.3. A *connection* on P is a smooth functor

$$\mathrm{hol} : PM \to \mathrm{Trans}(P)$$

where PM is the smooth category (groupoid!) of equivalence classes of paths in M, such that

$$\mathrm{hol}(x) = P_x \ \forall x \in M$$

[figure: base of M, with $\gamma : x \to y$ in it below; and tubular space above, with circular cross sections $\mathrm{hol}(x) = P_x, \mathrm{hol}(y) = P_y$ above x, y respectively; $\mathrm{hol}(\gamma) : P_x \to P_y$ by functoriality]

We can think of any smooth space M as a smooth category with

- points of M as objects: $\mathrm{Ob}(M) = M$; and
- only identity morphisms: $\mathrm{Mor}(M) = M$.

Then the condition $\mathrm{hol}(x) = P_x$ says:

$$\begin{tikzcd}
PM \arrow[rr, "\mathrm{hol}"] & & \mathrm{Trans}(P) \\
& M \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commutes.

6.3. **Getting back smooth circle-valued functors.** Let's examine this locally: for any $x \in M$, there's an open set $U \ni x$ and an isomorphism γ

such that

$$\begin{tikzcd} P|_U := \pi^{-1}(U) \arrow[rr, "\gamma", "\sim"'] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

commutes.

The inclusion $U \hookrightarrow M$ gives a smooth functor

$$PU \to PM$$

and a smooth functor

$$\mathrm{Trans}(P|_U) \to \mathrm{Trans}(P)$$

In fact, we get a commuting prism

$$\begin{tikzcd} & & & PM \arrow[r, "\mathrm{hol}"] & \mathrm{Trans}(P) \\ P|_U := \pi^{-1}(U) \arrow[urrr] \arrow[r, "\mathrm{hol}|_U"] & \mathrm{Trans}(P|_U) \arrow[urrr] & & M \arrow[u] \arrow[ur] & \\ & U \arrow[ul] \arrow[u] \arrow[urr] & & & \end{tikzcd}$$

for a unique $\mathrm{hol}|_U : PU \to \mathrm{Trans}(P|_U)$. So we can "restrict parallel transport to U". But we have (as above)

$$\begin{tikzcd} P|_U \arrow[rr, "\gamma", "\sim"'] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

so $\mathrm{Trans}(P|_U) \cong \mathrm{Trans}(U \times G)$, where $U \times G$ is the trivial principal G-bundle over U. But $\mathrm{Trans}(U \times G)$ is easy to understand:

[figure: cylinder called $U \times G$ over U, and $\gamma : x \to y$ gives some rotation by g]

$$\begin{aligned} \mathrm{Ob}(\mathrm{Trans}(U \times G)) &\cong U \\ \mathrm{Mor}(\mathrm{Trans}(U \times G)) &\cong U^2 \times G \end{aligned}$$

and in fact, $\mathrm{Trans}(U \times G)$ is a product of categories:

$$\mathrm{Trans}(U \times G) \cong \mathrm{Codisc}(U) \times G$$

where $\mathrm{Codisc}(U)$ has elements $x \in U$ as objects, and 1-morphisms from any object to any other - and G is regarded as a 1-object category. So we get

$$PU \stackrel{\mathrm{hol}|_U}{\longrightarrow} \mathrm{Trans}(P|_U) \cong \mathrm{Trans}(U \times G) \cong \mathrm{Codisc}(U) \times G \stackrel{\pi_2}{\longrightarrow} G$$

which is a smooth functor from PU to G which contains all the information in $\mathrm{hol}|_U$. But...

Theorem 6.4. *Smooth functors* hol : $PU \to G$ *are in one-to-one correspondence with* $\mathfrak{g}$*-valued 1-forms, where* $\mathfrak{g}$ *is the Lie algebra of* G*.*

So connections on principal G-bundles are locally described by $\mathfrak{g}$-valued 1-forms. For $G = U(1)$, these are just 1-forms.

7. May 15, 2007: Three approaches to connections - and smooth anafunctors

Last time we described a connection on a principal G-bundle $P \to M$ as a smooth functor:

$$\mathrm{hol} : PM \to \mathrm{Trans}(P)$$

such that $\mathrm{hol}(x) = P_x$. Last time, we expressed this last clause by saying that

$$\begin{tikzcd} & \mathrm{Disc}(M) \arrow[dl] \arrow[dr] & \\ PM \arrow[rr, "\mathrm{hol}"] & & \mathrm{Trans}(P) \end{tikzcd}$$

commutes, where $\mathrm{Disc}(M)$ is the *discrete category on* M. In other words, we're looking at smooth categories and functors *under* $\mathrm{Disc}(M)$.

A better way to express this clause is to say that

$$\begin{tikzcd} PM \arrow[rr, "\mathrm{hol}"] \arrow[dr] & & \mathrm{Trans}(P) \arrow[dl] \\ & \mathrm{Codisc}(M) & \end{tikzcd}$$

commutes, where $\mathrm{Codisc}(M)$ is the codiscrete category on M.

Okay... now for a big chart...

G-bundles	Connections	Gauge Transformations
Trivial G-bundles $M \times G \to M$	smooth functors $\mathrm{hol} : PM \to G$, or equivalently, $\mathfrak{g}$-valued 1-forms on M	smooth natural transformations $\begin{tikzcd} PM \arrow[r, bend left, "\mathrm{hol}_1"] \arrow[r, bend right, "\mathrm{hol}_2"'] \arrow[r, phantom, "\Downarrow g"] & G \end{tikzcd}$ or equivalently, smooth functions $g : M \to G$ with $\mathrm{hol}_2(\gamma) = g_y \, \mathrm{hol}_1(\gamma) g_x^{-1}$ for $\gamma : x \to y$ in PM.
Any fixed G-bundle $P \to M$	smooth functors $\begin{tikzcd} PM \arrow[r, "\mathrm{hol}"] \arrow[dr] & \mathrm{Trans}(P) \arrow[d] \\ & \mathrm{Codisc}(M) \end{tikzcd}$	smooth natural transformations $\begin{tikzcd} PM \arrow[rr, bend left, "\mathrm{hol}_1"] \arrow[rr, bend right, "\mathrm{hol}_2"'] \arrow[rr, phantom, "\Downarrow g"] \arrow[dr] & & G \arrow[dl] \\ & \mathrm{Codisc}(M) & \end{tikzcd}$
A variable G-bundle over M	smooth anafunctors $\mathrm{hol} : PM \to G$	smooth ananatural transformations $\begin{tikzcd} PM \arrow[r, bend left, "\mathrm{hol}_1"] \arrow[r, bend right, "\mathrm{hol}_2"'] \arrow[r, phantom, "\Downarrow g"] & G \end{tikzcd}$

Remark 7.1.

(1) Connections mod gauge transformations are classified by various forms of cohomology. For example,

$$\{U(1)\text{-bundles with connections over } M\}/\langle \text{gauge transformations} \rangle$$

is an example of Deligne cohomology.

(2) To get a smooth functor, "parallel transport", or

$$\mathrm{hol} : PM \to G$$

from a $\mathfrak{g}$-valued 1-form A on M, we set

$$\mathrm{hol}(\gamma) = Pe^{\int_\gamma A}$$

If $G = U(1)$, then $\mathfrak{g} = \mathfrak{u}(1) = i\mathbb{R} \cong \mathbb{R}$, so a $\mathfrak{g}$-valued 1-form amounts to a 1-form A and then

$$\mathrm{hol}(\gamma) = e^{i\int_\gamma A}$$

(i.e., path-ordered exponentiation reduces to orginary exponentiation, since $U(1)$ is abelian).

(3) Given smooth functors

$$F_1, F_2 : \mathcal{C} \to \mathcal{D}$$

a *smooth natural transformation* is a smooth map

$$\alpha : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$$

such that for all morphisms $f : x \to y$ in $\mathcal{C}$, the following square exists

$$\begin{tikzcd} F_1(x) \arrow[r, "F_1(f)"] \arrow[d, "\alpha_x"'] & F_1(y) \arrow[d, "\alpha_y"'] \\ F_2(x) \arrow[r, "F_2(f)"] & F_2(y) \end{tikzcd}$$

and commutes.

So a gauge transformation

$$\begin{tikzcd} PM \arrow[r, bend left=50, "\mathrm{hol}_1"{name=U}] \arrow[r, bend right=50, "\mathrm{hol}_2"'{name=D}] & G \arrow[Rightarrow, from=U, to=D, "g"] \end{tikzcd}$$

is a smooth natural transformation from hol_1 to hol_2, i.e. a smooth map

$$g : \mathrm{Ob}(PM) \to \mathrm{Mor}(G)$$

i.e.

$$g : M \to G$$

such that given a path $\gamma : x \to y$ in M, this square commutes:

$$\begin{tikzcd} * \arrow[r, "\mathrm{hol}_1(\gamma)"] \arrow[d, "g_x"'] & * \arrow[d, "g_y"'] \\ * \arrow[r, "\mathrm{hol}_2(\gamma)"] & * \end{tikzcd}$$

where $*$ is the one object of our group G. This says that

$$\mathrm{hol}_2(\gamma) = g_y \,\mathrm{hol}_1(\gamma) g_x^{-1}$$

(4) Given any G-bundle $P \to M$ and two connections

$$\mathrm{hol}_1, \mathrm{hol}_2 : PM \to \mathrm{Trans}(P),$$

a smooth natural transformation

$$\begin{tikzcd} PM \arrow[r, bend left, "\mathrm{hol}_1"] \arrow[r, bend right, "\mathrm{hol}_2"'] \arrow[r, phantom, "\Downarrow g"] & \mathrm{Trans}(P) \end{tikzcd}$$

is a smooth map

$$g : \mathrm{Ob}(PM) \to \mathrm{Mor}(\mathrm{Trans}(P))$$

(here, $\mathrm{Ob}(PM) = M$) such that the following square commutes:

$$\begin{array}{ccc} P_x & \xrightarrow{\mathrm{hol}_1(\gamma)} & P_y \\ {\scriptstyle g_x}\downarrow & & \downarrow{\scriptstyle g_y} \\ P_x & \xrightarrow{\mathrm{hol}_2(\gamma)} & P_y \end{array}$$

[figure: cylinder above $\gamma : x \to y$ with cross sections P_x, P_y above x, y]

where $g_x : P_x \to P_x$ is a G-torsor morphism

$$g_x(ph) = g_x(p)h \ \forall p \in P_x, h \in G$$

and similarly for g_y.

(5) What's a smooth anafunctor? Given smooth categories $\mathcal{C}$ and $\mathcal{D}$, a smooth anafunctor is the *right* kind of thing going from $\mathcal{C}$ to $\mathcal{D}$, generalizing a smooth functor. A smooth anafunctor "looks locally like a smooth functor", so it can be thought of as a functor which is locally isomorphic (via natural isomorphisms) to a smooth one. More precisely:

Definition 7.2. Let $\mathcal{C}, \mathcal{D}$ be smooth categories. A *smooth anafunctor* $F : \mathcal{C} \to \mathcal{D}$ consists of

(a) an open cover $\{U_\alpha\}$ of $\mathrm{Ob}(\mathcal{C})$, where $\mathrm{Ob}(\mathcal{C})$ is a topological space with the finest topology such that every plot in $\mathrm{Ob}(\mathcal{C})$ is continuous.

(b) smooth functors $F_\alpha : \mathcal{C}_\alpha \to \mathcal{D}$, where $\mathcal{C}_\alpha$ has
- objects in U_α as objects
- morphisms between these as morphisms.

(c) smooth natural isomorphisms

$$\begin{tikzcd} & \mathcal{C}_\alpha \arrow[dr, "F_\alpha"] \arrow[dd, phantom, "\Downarrow g_{\alpha\beta}"] & \\ \mathcal{C}_{\alpha\beta} \arrow[ur] \arrow[dr] & & \mathcal{D} \\ & \mathcal{C}_\beta \arrow[ur, "F_\beta"'] & \end{tikzcd}$$

where $\mathcal{C}_{\alpha\beta} \hookrightarrow \mathcal{C}_\alpha, \mathcal{C}_\beta$ has
- objects in $U_{\alpha\beta} := U_\alpha \cap U_\beta$ as objects;
- morphisms between these as morphisms.

(d) the "cocycle conditions" that are finally requred:

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$$

on objects in $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$.

8. May 22, 2007: Bundles, connections, cohomology, and anafunctors

Last time JB claimed that if M is a smooth space (e.g. a manifold), and G a Lie group, then principal G-bundles with connection over M correspond to smooth anafunctors

$$\mathrm{hol}: PM \to G$$

JB also claimed that isomorphisms between principal G-bundles with connection, correspond to smooth ananatural transformations

$$\begin{tikzcd} PM \arrow[r, bend left=50, "\mathrm{hol}"{name=U}] \arrow[r, bend right=50, "\mathrm{hol}'"'{name=D}] & G \arrow[from=U, to=D, Rightarrow, "g"] \end{tikzcd}$$

8.1. **The picture without connection.** If we leave out the connection, we get a simpler version of this story. Principal G-bundles over M correspond to smooth anafunctors

$$\mathrm{hol}: \mathrm{Disc}(M) \to G$$

and isomorphisms between principal G-bundles correspond to smooth ananatural transformations

$$\begin{tikzcd} \mathrm{Disc}(M) \arrow[r, bend left=50, "\mathrm{hol}"{name=U}] \arrow[r, bend right=50, "\mathrm{hol}'"'{name=D}] & G \arrow[from=U, to=D, Rightarrow, "g"] \end{tikzcd}$$

Let's see why this simpler version works. Our ultimate goal is to show

$$\frac{\{\text{Principal } G\text{-bundles over } M\}}{\{\text{isomorphisms}\}} \cong \frac{\langle \text{smooth anafunctors} : \mathrm{Disc}(M) \to G \rangle}{\langle \text{ananatural transformations} \rangle} \cong \check{H}^1(M, G)$$

where $\check{H}^1(M, G)$ is the first Čech cohomology of M with coefficients in G. A famous example:

$$\check{H}^1(M, U(1)) \cong H^2(M, \mathbb{Z})$$

where $H^2(M, \mathbb{Z})$ is the second cohomology of M with coefficients in $\mathbb{Z}$.

For example, the sphere

[figure: sphere $M = S^2$]

has $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ (S^2 has "one 2-dimensional hole"). So, (isomorphism classes of) principal $U(1)$-bundles over S^2 are classified by an integer, the "first Chern class" c_1.

8.2. **A first look at Čech cohomology.** Suppose $P \to M$ is a principal G-bundle. It's locally trivial: for each point $x \in M$ there's an open neighbourhood $U \ni x$ for which we have an isomorphism of G-spaces:

$$\begin{tikzcd} P|_U := \pi^{-1}(U) \arrow[rr, "\gamma", "\sim"'] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

Let's go ahead and pick an open cover $\{U_i : i \in I\}$ of M, and isomorphisms

$$\begin{tikzcd} P|_{U_i} := \pi^{-1}(U_i) \arrow[rr, "\gamma_i", "\sim"'] \arrow[dr, "\pi"'] & & U_i \times G \arrow[dl, "\pi_1"] \\ & U_i & \end{tikzcd}$$

Let $U_{ij} = U_i \cap U_j$; we get "transition functions"

$$g_{ij} := \gamma_i \circ \gamma_j^{-1} : U_{ij} \times G \xrightarrow{\sim} U_{ij} \times G$$

(here, γ_i means γ_i restricted to $P|_{U_{ij}}$; similarly for γ_j). This is an isomorphism of right G-spaces, necessarily given by left multiplication by some G-valued function on U_{ij}.

By abuse of notation, we also call this function

$$g_{ij} : U_{ij} \to G$$

This is a "Čech 1-cochain": in general a Čech n-cochain would be a bunch of maps

$$g_{i_0,\ldots,i_n} : U_{i_0,\ldots,i_n} = \bigcap_{j=0}^{n} U_{i_j} \to G$$

So: any principal G-bundle over M gives a Čech 1-cochain. In fact, g_{ij} satisfies an equation:

$$g_{ij} \cdot g_{jk} = \gamma_i \gamma_j^{-1} \circ \gamma_j \gamma_k^{-1} = \gamma_i \gamma_k^{-1} = g_{ik}$$

on $U_{ijk} := U_i \cap U_j \cap U_k$. This is called the cocycle condition, and we say g_{ij} is a *Čech 1-cocycle*. The cocycle condition really says that this 2-cochain

$$g_{ij} g_{jk} g_{ik}^{-1} : U_{ijk} \to G$$

is trivial, i.e. it equals 1.

Conversely, given a Čech 1-cocycle, you can build a principal G-bundle over M. So:

principal G-bundles correspond to Čech 1-cocycles.

What do isomorphisms of principal G-bundles correspond to?

8.3. **Quotienting out the isomorphisms.** Suppose P, P' are two principal G-bundles over M, and we have an isomorphism

$$\begin{tikzcd} P \arrow[rr, "f", "\sim"'] \arrow[dr] & & P' \arrow[dl] \\ & M & \end{tikzcd}$$

(i.e. f is a smooth map of right G-spaces making the triangle commute, where we can think of M as a trivial G-space - so this diagram lives in the category of smooth G-spaces). We can find an open cover $\{U_i : i \in I\}$ of M such that $P|_{U_i}$ and $P'|_{U_i}$ are trivializable:

$$U_i \times G \xrightarrow{\gamma_i^{-1}} P|_{U_i} \xrightarrow{f} P'|_{U_i} \xrightarrow{\gamma_i'} U_i \times G$$

(Over here, all maps are isomorphisms, and γ_i, γ_i' are trivializations - isomorophisms of right G-spaces over U_i.) Let

$$f_i := \gamma_i' \circ f \circ \gamma_i^{-1}$$

for $i \in I$ be the composite; these f_i's describe our bundle isomorphism f locally, i.e. we can reconstruct $f : P|_{U_i} \to P'|_{U_i}$ from f_i, γ_i, γ_i'. Since $f_i : U_i \times G \to U_i \times G$ is an isomorphism of G-bundles, it's determined by a function from U_i to G, which we also call f_i: we can write

$$f_i(x, g) = (x, f_i(x, g))$$

for some $f_i : U_i \to G$.

Recall that a bunch of maps $\{f_i : U_i \to G : i \in I\}$ is called a *Čech 0-cochain.* The bundle P is described by a Čech 1-cochain:

$$g_{ij} := \gamma_i \gamma_j^{-1}$$

and similarly for $P' : g_{ij}' = \gamma_i'(\gamma_j')^{-1}$, where $\gamma_{ij}, \gamma_{ij}' : U_{ij} \to G$. In fact, g_{ij} and g_{ij}' are related by the f_i's as follows:

$$f_i g_{ij} = g_{ij}' f_j$$

Let's čeck this! We indeed have

$$\gamma_i' f \gamma_i^{-1} \circ \gamma_i \gamma_j^{-1} = \gamma_i'(\gamma_j')^{-1} \circ \gamma_j' f \gamma_j^{-1}$$

because both sides equal $\gamma_i' f \gamma_j^{-1}$.

In short, the "difference" of the two Čech 1-cochains (actually 1-cocycles) g_{ij} and g_{ij}' is given by the Čech 0-cochain f_i

Remark 8.1. But if G is nonabelian, we *cannot* actually write $g_{ij}(g_{ij}')^{-1} = f_j f_i^{-1}$. This works only if G is abelian. We're studying "nonabelian Čech cohomology"!

So:

$$\frac{\{G\text{-bundles } P \to M \text{ such that } P|_{U_i} \text{ are trivial}\}}{\langle\text{isomorphisms}\rangle} \cong \frac{\{\text{Čech 1-cocycles } g_{ij} : U_{ij} \to G\}}{\langle g_{ij} \sim g'_{ij} \text{ iff } f_i g_{ij} = g'_{ij} f_j\rangle}$$

Taking the "limit" (really the colimit!) as the cover $\{U_i\}$ gets finer, we get

$$\frac{\{G\text{-bundles } P \to M\}}{\langle\text{isomorphisms}\rangle} = \lim_{\overrightarrow{U}} \frac{\{\text{Čech 1-cocycles } g_{ij} : U_{ij} \to G\}}{\sim}$$

and the right-hand side is defined to be $\check{H}^1(M, G)$, the first Čech cohomology of M with coefficients in G!

Remark 8.2. There's another description of this (due to Toby Bartels):

$$\check{H}^1(M, G) = \frac{\{\text{smooth anafunctors } F : \mathrm{Disc}(M) \to G\}}{\langle\text{ananatural isomorphisms}\rangle}$$

9. May 29, 2007: Čech cohomology and ananatural category theory

9.1. Definitions - old and new. Recall: given smooth categories $\mathcal{C}, \mathcal{D}$, a smooth anafunctor $F : \mathcal{C} \to \mathcal{D}$ consists of

(1) an open cover $\{U_i : i \in I\}$ of $\mathrm{Ob}(\mathcal{C})$
(2) smooth functors $F_i : \mathcal{C}|_{U_i} \to \mathcal{D}$
(3) smooth natural isomorphisms

$$\begin{tikzcd} \mathcal{C}|_{U_{ij}} \arrow[r, bend left, "F_j"] \arrow[r, bend right, "F_i"'] \arrow[r, phantom, "\Downarrow g_{ij}"] & \mathcal{D} \end{tikzcd}$$

such that

(4) the cocycle condition

$$g_{ij} g_{jk} = g_{ik}$$

holds on $\mathcal{C}|_{U_{ijk}}$.

So - what's a smooth anafunctor

$$F : \mathrm{Disc}(M) \to G?$$

Well, it must consist of

(1) an open cover $\{U_i : i \in I\}$ of $\mathrm{Ob}(\mathrm{Disc}(M)) = M$
(2) smooth functors $F_i : \mathrm{Disc}(U_i) \to G$ - but there's only one, since $\mathrm{Disc}(U_i)$ is discrete, and G has only one object.
(3) smooth natural isomorphisms

$$\begin{tikzcd} \mathrm{Disc}(U_{ij}) \arrow[r, bend left, "F_j"] \arrow[r, bend right, "F_i"'] \arrow[r, phantom, "\Downarrow g_{ij}"] & G \end{tikzcd}$$

For each object in $\mathrm{Disc}(U_{ij})$, i.e. $x \in U_{ij}$, this gives an isomorphism $g_{ij}(x) : F_j(x) = * \to * = F_i(x)$, where $* \in G$ is unique.

In short, $g_{ij} \in G$. Why is g_{ij} natural? Well,

$$\begin{tikzcd} F_j(x) \arrow[r, "F_j(1_x)"] \arrow[d, "g_{ij}(x)"'] & F_j(x) \arrow[d, "g_{ij}(x)"'] \\ F_i(x) \arrow[r, "F_i(1_x)"] & F_i(x) \end{tikzcd}$$

must commute, and obviously does!

(4) And we require $g_{ij} g_{jk} = g_{ik}$.

All this is just a Čech 1-cocycle! So:

$$\{\text{smooth anafunctors } F : \mathrm{Disc}(M) \to G\} \cong \{\text{Čech 1-cocycles}\}$$

and we can cook up a definition of "ananatural transformation", such that

$$\frac{\{\text{smooth anafunctors } F : \mathrm{Disc}(M) \to G\}}{\langle \text{ananatural transformations} \rangle} \cong \check{H}^1(M, G)$$

Here is the definition in general.

Definition 9.1. Given smooth anafunctors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] & \mathcal{D} \end{tikzcd},$$

a *smooth ananatural transformation*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow f"] & \mathcal{D} \end{tikzcd}$$

is a collection of smooth natural transformations

$$\begin{tikzcd} \mathcal{C}|_{U_i} \arrow[r, bend left, "F_i"] \arrow[r, bend right, "F'_i"'] \arrow[r, phantom, "\Downarrow f_i"] & \mathcal{D} \end{tikzcd},$$

such that

$$f_i g_{ij} = g'_{ij} f_j \ \forall i, j$$

where g_{ij}, g'_{ij} come from the anafunctors F, F', in the way we described (above).

9.2. **One possible generalization.** In summary:

$$\frac{\{G\text{-bundles over } M\}}{\langle \text{isomorphisms} \rangle} \cong \check{H}^1(M, G) \cong \frac{\{\text{smooth anafunctors } F : \mathrm{Disc}(M) \to G\}}{\langle \text{ananatural isomorphisms} \rangle}$$

but in fact:

$$\frac{\{G\text{-bundles with connection over } M\}}{\langle \text{isomorphisms} \rangle} \cong \frac{\{\text{smooth anafunctors } \mathrm{hol} : PM \to G\}}{\langle \text{ananatural isomorphisms} \rangle}$$

and this deserves to be called "$\check{H}^1(PM, G)$" - now we're really doing Čech cohomology of a smooth category! More generally, given two smooth 1-categories $\mathcal{C}, \mathcal{D}$, let's define

$$\check{H}^1(\mathcal{C}, \mathcal{D}) := \frac{\{\text{smooth anafunctors } F : \mathcal{C} \to \mathcal{D}\}}{\langle \text{ananatural isomorphisms} \rangle}$$

Then replace 1 by "n"!

10. June 5, 2007: Review and prospectus

10.1. **Phases and functors.** We've been trying to understand classical and quantum physics in a unified way, starting with a category $\mathcal{C}$ of "configurations" (ways things can be) and "processes" (ways things can become), with a functor

$$S : \mathcal{C} \to \mathbb{R}$$

or

$$\Phi : \mathcal{C} \to U(1)$$

assigning to any process $\gamma : x \to y$ its "action" $S(\gamma) \in \mathbb{R}$ or "phase" $\Phi(\gamma) \in U(1)$. These are often related by

$$\Phi(\gamma) = e^{iS(\gamma)}$$

but Φ need not be of this form.

Classical	Quantum
$\mathcal{C}$ is a smooth category.	$\mathcal{C}$ is a measurable category.
$\Phi : \mathcal{C} \to U(1)$ is a smooth functor.	$\Phi : \mathcal{C} \to U(1)$ is a measurable functor.
Given $x, y \in \mathcal{C}$, find $\gamma : x \to y$ with $\delta\Phi(\gamma) = 0$.	Given $x, y \in \mathcal{C}$, find $\langle y, x\rangle = \displaystyle\int_{\gamma : x \to y} \Phi(\gamma)\ \mathscr{D}\gamma \in \mathbb{C}$.

We can try to unify these using *rigs*, but in both cases there are subtleties that complicate the picture above. In the quantum case, we need to stretch the theory of integration beyond traditional measure theory, in order to do the "path integral" above. In the classical case, smooth functors aren't enough - we need smooth anafunctors. So, there's a lot more to be understood, both classically and quantum-ly.

Look at the classical side of the prospectus. In classical mechanis, we often start with a phase space (X, ω) - a symplectic manifold. From this, we build a smooth category

$$\mathcal{C} = PX,$$

the path groupoid. Then we seek a smooth anafunctor

$$\Phi : PX \to U(1)$$

such that if γ is a loop that bounds a surface Σ (i.e. $\partial\Sigma = \gamma$)

[figure: surface Σ, possibly with handles, bounded by (oriented) γ]

then we have

$$\Phi(\gamma) = e^{i \int_\Sigma \omega}$$

There could be many Φ's, or no Φ's, that do this. For such a Φ to exist, we need:

$$\partial\Sigma = \emptyset \Rightarrow \int_\Sigma \omega \in 2\pi\mathbb{Z}$$

[figure: Σ with a point boundary x - so $\gamma = 1_x$]

since we need $\Phi(1_x) = 1$. We say a closed 2-form is *integral* if $\int_\Sigma \omega \in 2\pi\mathbb{Z}$ for all closed $\Sigma \subset X$. Conversely, if ω is integral, we can indeed find Φ as desired. If ω is integral,

$$[\omega] \in H^2(X, \mathbb{Z}) \cong \check{H}^1(X, U(1))$$

and we've seen that $\check{H}^1(X, U(1))$ classifies principal $U(1)$-bundles over X. In fact, our smooth anafunctor

$$\Phi : PX \to U(1)$$

arises as follows: pick a $U(1)$-bundle $P \to X$ corresponding to $[\omega]$, pick a connection A on it, and let

$$\Phi(\gamma) = \mathrm{hol}(\gamma)$$

where holonomy is defined using A.

To go further, we should use geometric quantization to get a Hilbert space from $(X, \omega, P \to X, A)$ - this requires extra structure on X, e.g. a Kähler structure. It would be great to show that these Hilbert spaces match those given by our path integral procedure!

10.2. **From particles to strings: categorification.** Also, in this course we *hinted* at how to categorify all this stuff, to go from particle physics to string physics. Naively, we could categorify our previous chart:

Classical	Quantum
$\mathcal{C}$ is a smooth 2-category.	$\mathcal{C}$ is a measurable 2-category.
$\Phi : \mathcal{C} \to U(1) - \mathrm{Tor} \cong U(1)[1]$ is a smooth functor.	$\Phi : \mathcal{C} \to U(1)[1]$ is a measurable functor.
Given $\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_2"'] & y \end{tikzcd}$, find $\Sigma :$ $\gamma_1 \Rightarrow \gamma_2$ such that $\delta\Phi(\Sigma) = 0$.	Given $\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_2"'] & y \end{tikzcd}$, we can compute an amplitude $\langle \gamma_2, \gamma_1 \rangle = \int_{\Sigma : \gamma_1 \Rightarrow \gamma_2} \Phi(\Sigma)\ \mathscr{D}\Sigma.$

The path integrals become a lot harder now, but are still manageable. Over on the classical side, we really need 2-anafunctors.

Urs Schreiber and JB showed:

Given a smooth space X, there's a smooth 2-groupoid P_2X, where:

$\bullet$ objects are points of X;

$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \end{tikzcd}$ morphisms are smooth paths in X; and

$\begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$ 2-morphisms are thin homotopy classes of paths-of-paths in X

Given a "smooth 2-group" G (e.g. $U(1)[1]$), 2-connections on principal G-2-bundles over X correspond to smooth 2-anafunctors

$$\mathrm{hol} : P_2X \to G$$

So - everything we said about bundles and connections categorifies!

In string theory, there are nice examples of smooth 2-groups. For example, given any compact simple (simply connected) Lie group G, there's a 2-group "String_H". When $H = \mathrm{Spin}(n)$ is the double cover of $SO(n)$, "String_H" is the symmetry 2-group involved in studying "spinning strings", and basic in elliptic cohomology, as studied by Stolz and Teichner.