Cohomology & the Category of Simplices

Why are simplices better (in many ways) than globes:

or cubes:

One answer is that a simplex

is a special sort of category — namely, a finite totally ordered set! The set is the set of vertices; the $<$ relation gives the edges. Every finite totally ordered set is isomorphic to a finite ordinal:

- $0 = \{\}$
- $1 = \{0\}$
- $2 = \{0,1\}$
- $3 = \{0,1,2\}$

where $<$ is now $\in$ (or $\subseteq$)
There's a category $\Delta_{\text{alg}}$ - the algebraist's category of simplices - where the objects are finite totally ordered sets (or simplices) & the morphisms are order preserving functions $f: S \to T$, i.e. functions with

$$x \leq y \implies f(x) \leq f(y)$$

(If we think of simplices as (toset) categories, order preserving functions are just functors.) A typical morphism in $\Delta_{\text{alg}}$ looks like:

\[
\begin{align*}
\{ 0, 1, 2, 3, 4, 5, 6 \} &= 6 \\
\{ 0, 1, 2, 3, 4, 5, 6 \} &= 7
\end{align*}
\]

$\Delta_{\text{alg}}$ is generated by the objects $0, 1, 2, \ldots$ and certain special morphisms like:

\[
\begin{align*}
d_2 &
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 1 & 2 & 3 & 4
\end{array}
\end{align*}
\]

1-1 and almost onto

Or more generally we need

$$d_j: n \to n+1 \quad 0 \leq j \leq n$$

- the order preserving map whose image only fails to contain $j$. 

We also need ones like:

\[\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 1 & 2 & 3 \\
\end{array}\]

onto and almost 1-1

or more generally

\[e_j : n+1 \to n \quad 0 \leq j \leq n-1\]

- the order preserving map for which \(j\) is the only element with two preimages. By "generated" we mean any order preserving function \(f : n \to m\) is a composite of maps \(d_j, e_j\). E.g.

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow \\
\bullet & \bullet & \bullet \\
\end{array} = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow \\
\bullet & \bullet & \bullet \\
\end{array}
\]

In terms of topology, \(d_j : n \to n+1\) is a face map:

\[\begin{array}{cc}
\begin{array}{ccc}
0 & 1 & 2 \\
\end{array} \xrightarrow{d_0} \begin{array}{ccc}
0 & 1 & 2 \\
\end{array}
\end{array}\]

\[d_0(0) = 1, \quad d_0(1) = 2, \quad d_0(2) = 3\]

face that lacks the 0th vertex.
Similarly \( e_j : n+1 \to n \) is a degeneracy map

\[
\begin{align*}
\begin{array}{c}
0 \\
1 \\
2 \\
3
\end{array}
\end{align*}
\quad \xrightarrow{e_j} \quad
\begin{align*}
\begin{array}{c}
0 \\
1 \\
2
\end{array}
\end{align*}
\]

\( e_j \) squashes vertices \( j \) & \( j+1 \) down to the vertex \( j \).

\[ e_j(0) = 0, \quad e_j(1) = 1, \quad e_j(2) = 1, \quad e_j(3) = 2 \]

We can turn \( n \) into a space, the standard \((n-1)\)-simplex:

\[
\Delta_{n-1} = \left\{ (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n : x_i \geq 0 \quad \text{and} \quad \sum x_i = 0 \right\}
\]

Then any morphism \( f : n \to m \) can be turned into an affine map

\[
\Delta_{f_{-1}} : \Delta_{n-1} \to \Delta_{m-1}
\]

(a silly name (but a systematic one, since \( \Delta_{-1} \) is a functor!)

---

the unique affine map sending the \( j \)th vertex of \( \Delta_{n-1} \) to the \( f(j) \)-th vertex of \( \Delta_{m-1} \).
This process of turning ordinals into spaces & order-preserving maps into continuous maps is a functor:

\[ \Delta_{-1} : \Delta_{\text{alg}} \rightarrow \text{Top} \]

When \( n = 0 \), this functor gives "the standard \(-1\)-simplex" which is \( \emptyset \in \mathbb{R}^0 \). If this freaks us out, we restrict attention to \( \Delta_{\text{top}} \), the category of nonempty totally ordered sets & order-preserving functions. This is the topologist's category of simplices. Topologists call this \( \Delta \) and call \( \Delta_{\text{alg}} \) the augmented category of simplices.

**Def:** A simplicial set is a functor

\[ F : \Delta_{\text{top}}^{\text{op}} \rightarrow \text{Set} \]

**Claim:** such a thing looks sort of like:

![Diagram](image)
\( \Delta_{\text{top}} \) looks sort of like:

\[
\begin{array}{ccc}
\bullet & \xrightarrow{d_0} & \\
1 & \xleftarrow{e_0} & \bullet \xrightarrow{d_1} 2 \\
& \xleftarrow{d_1} & \bullet & \xrightarrow{e_1} 3 \\
& & \triangle & \ldots
\end{array}
\]

\( \Delta_{\text{top}}^{\text{op}} \) is just the same diagram with all the arrows reversed.

So a simplicial set \( F \) looks like

\[
\begin{array}{ccc}
F(1) & \xrightarrow{F(d_0)} & F(2) \\
\wedge & \xleftarrow{F(e_0)} & \wedge \\
\downarrow & \xleftarrow{F(d_1)} & \downarrow \\
\text{set of vertices} & \text{set of edges} & \text{set of triangles}
\end{array}
\]

Given any edge \( \xrightarrow{x} \), we get two vertices:

\[
\begin{array}{ccc}
F(d_0)(x) & F(d_0)(x) \\
\bullet & \bullet \\
\xrightarrow{x} & \xleftarrow{x}
\end{array}
\]

Similarly:

\[
\begin{array}{ccc}
1 & \xrightarrow{F(d_0)(x)} & \bullet \\
& \xleftarrow{F(d_0)(x)} & 2 \\
& & \triangle
\end{array}
\]
Usually people write

\[ \partial_j = F(d_j) \]
\[ \varepsilon_j = F(e_j) \]

So a simplicial set is like

\[
\begin{array}{ccccc}
F(1) & \xrightarrow{\varepsilon_0} & F(2) & \xleftarrow{\partial_0} & F(3) \\
\downarrow{\varepsilon_1} & & \downarrow{\partial_1} & & \downarrow{\varepsilon_2} \\
\end{array}
\]

The way cohomology of spaces works:

\[
\text{Top} \xrightarrow{S\text{nerve}} \text{hom}(\Delta^{op}, \text{Set}) \xrightarrow{\text{Fun-}} \text{hom}(\Delta^{op}, \text{AbGp}) \cong \text{Chain complexes}
\]

- Free abelian group
- Simplicial abelian group
- Dold-Kan Theorem