Chain Complexes

We can get simplicial sets from topological spaces, algebraic gadgets, etc. To count the holes (of various dimensions) of a simplicial set, we process it:

\[
\begin{align*}
\text{[Simplicial sets]} & \quad \downarrow \text{compose with free abelian group} \downarrow \text{factor } \text{Set \to AbGrp} \\
& \quad \text{(i.e. take formal } \mathbb{Z}-\text{linear combinations)} \\
\text{[Simplicial abelian groups]} & \quad \downarrow \text{Dold-Kan theorem} \\
\text{[chain complexes]} & \quad \downarrow \text{take the homology} \\
\text{[graded abelian groups]} \\
\end{align*}
\]

Just for now, let's write

\[
\Delta = \Delta_{\text{top}}
\]

So

\[
\Delta = 1 \circlearrowright 2 \circlearrowright 3 \cdots
\]
But "n" corresponds to the (n-1)-simplex, so let's reindex and write:

\[
\triangle = \begin{array}{cccc}
& 0 & \circlearrowright & \circlearrowright & z & \cdots \\
\text{the 0-simplex} & \text{the 1-simplex} & \text{the 2-simplex} & \triangle
\end{array}
\]

For a topologist, a simplicial set is a functor

\[ X : \Delta^{op} \to \text{Set} \]

i.e., a diagram like

\[
\begin{array}{ccc}
X(0) & \xrightarrow{\partial_0} & X(1) \\
\downarrow{\partial_2} & & \downarrow{\partial_2} \\
X(2) & & \cdots
\end{array}
\]

in \text{Set}.

To apply linear algebra, we take formal \(\mathbb{Z}\)-linear combinations of simplices & get abelian groups \(FX(n)\) where

\[ F : \text{Set} \to \text{AbGp} \]

is the free abelian group functor.
So we get

\[ F_X : \Delta^{op} \to \text{AbGr} \]

i.e. a diagram like

\[ \begin{array}{ccc}
F_X(0) & \xrightarrow{F_X(1)} & F_X(1) \\
\downarrow{F_X(2)} & & \downarrow{F_X(3)} \\
F_X(3) & \xleftarrow{F_X(2)} & F_X(2)
\end{array} \]

in \text{AbGr}.

In general, for any category \( C \), we call a functor

\[ G : \Delta^{op} \to C \]

a simplicial object in \( C \). In particular, we call a functor

\[ \Delta^{op} \to \text{AbGr} \]

a simplicial abelian group. In fact, a simplicial abelian group is just a chain complex of abelian groups: a sequence of abelian group homomorphisms

\[ C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} \cdots \]

such that \( \partial^2 = 0 \).

How do we turn a simplicial abelian group

\[ G : \Delta^{op} \to \text{AbGr} \]
into a chain complex Ch(G):

\[ Ch(G)_0 \xleftarrow{\partial} Ch(G)_1 \xleftarrow{\partial} Ch(G)_2 \xleftarrow{\partial} \cdots \]

Here's how: let

\[ Ch(G)_n = G(n) / \text{im } z_0 + \cdots + \text{im } z_{n-1} \]

where the \( z_i \)'s are degeneracies:

\[ G(0) \xrightarrow{z_0} G(1) \xrightarrow{z_1} G(2) \rightarrow \cdots \]

This lets us ignore degenerate simplices. Then, let

\[ \partial : Ch(G)_n \rightarrow Ch(G)_{n-1} \]

be given by

\[ \partial = \sum_{i=0}^{n} (-1)^i \partial_i \]

You need to check that:

1) \( \partial \) is well defined as a map \( Ch(G)_n \rightarrow Ch(G)_{n-1} \).
   (Check if \( x \in G_n \) is degenerate, \( \partial x = 0 \) mod degenerate simplices)

2) \( \partial^2 = 0 \).

For example:

\[ X \text{ is a simplicial set with } X(0) = \{x, y, z\} \]

\[ X(1) = \{f, g, h, s(x), s(y), s(z), t(x), t(y), t(z), 1, 2, 3\} \]

\[ X(2) = \text{degenerate 2-simplices} \]
This gives a simplicial abelian group $G = FX$ with

\[ G(0) = \mathbb{F}\{x, y, z\} \]
\[ G(1) = \mathbb{F}\{f, g, h, t_0(x), t_0(y), t_0(z)\} \]
\[ G(2) = \mathbb{F}\{ \text{degenerate 2-simplices} \} \]

This gives a chain complex $\text{Ch}(G)$ with

\[ \text{Ch}(G)_0 = \mathbb{F}\{x, y, z\} \]
\[ \text{Ch}(G)_1 = \mathbb{F}\{f, g, h\} \]
\[ \text{Ch}(G)_2 = \{0\} \]

and

\[ \partial : \text{Ch}(G)_1 \longrightarrow \text{Ch}(G)_0 \]

is given by

\[ \partial f = \partial_0 f - \partial_1 f = y - x \]
\[ \partial g = \partial_0 g - \partial_1 g = z - y \]
\[ \partial h = \partial_0 h - \partial_1 h = z - x \]

Here $\partial^2 = 0$ for trivial reasons. But here is still something interesting: $\partial^2 = 0$ means

\[ \text{im}(\partial : \text{Ch}(G)_{n+1} \rightarrow \text{Ch}(G)_n) \subseteq \ker(\partial : \text{Ch}(G)_n \rightarrow \text{Ch}(G)_{n-1}) \]

—if something is a boundary, it has vanishing boundary of its own. But not vice versa, since there can be "holes". So we can keep track of holes using the homology groups

\[ H_n(C) = \frac{\ker(\partial : \text{Ch}(G)_n \rightarrow \text{Ch}(G)_{n-1})}{\text{im}(\partial : \text{Ch}(G)_{n+1} \rightarrow \text{Ch}(G)_n)} \]
Our example has a hole:

\[ \text{ker}(d : C_1 \to C_0) = F\{f+g-h^3\} \]

So compare

\[ \text{im}(d : C_2 \to C_1) = \mathbb{E}0 \]

So

\[ H_1(C) = \frac{F\{f+g-h^3\}}{\mathbb{E}0} \cong \mathbb{Z} \]

We're getting the free abelian group on one generator, indicating that we have one "hole".

Now let's do a less trivial example by making \( X \) bigger. Our new \( X \) will look like this:

\[ \alpha, \beta \text{ are two 2-simplices filling in the same hole. This gives a higher dimensional hole:} \]
We calculate:

\[ \partial \alpha = f + g - h \]
\[ \partial \beta = f + g - h \]

So now

\[ \text{im} (\partial : C_2 \to C_1) = F \{ f + g - h \} \]
\[ \text{ker} (\partial : C_1 \to C_0) = F \{ f + g - h \} \]

so we still have \( \text{im} \leq \ker \) here, so

\[ C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \]

is zero: \( \partial^2 = 0 \). We also see that

\[ H_1(C) = \{0\} \]

we've filled the hole in the triangle. But

\[ \text{im} (\partial : C_3 \to C_2) = \{0\} \]
\[ \text{im} (\partial : C_2 \to C_1) = F \{ \alpha - \beta \} \]

so we get

\[ H_2(C) = \frac{F \{ \alpha - \beta \}}{\{0\}} \cong \mathbb{Z} \]

So here's one hole, but now of a higher dimension: \( \bullet \)

\( \alpha - \beta \) describes the surface of this 2-sphere, which has vanishing boundary but is not itself a boundary.