We've seen that in a weak $w$-category, associativity doesn't hold "on the nose." (in fact nothing holds "on the nose").

We've seen that given

\[ \begin{array}{ccc}
  \bullet & \bullet & \bullet \\
  a & b & c \\
\end{array} \quad a, b, c \in C \]

we get three kinds of "composite"

\[(ab)c \Leftrightarrow abc \Leftrightarrow a(bc)\]

These aren't equal, but we have a 2-morphism going between them.

They are related by 2-morphisms, "\(\Leftrightarrow\)" the "half associators" and their "weak inverses".

\[ \Rightarrow \circ \Leftarrow \]

Isn't the identity (there is a 3-morphism from the composite to the identity).

Pictorially: these 3 ways of composing are:

\[ \begin{array}{ccc}
  \bullet & \bullet & \bullet \\
  a & b & c \\
\end{array} \quad \begin{array}{ccc}
  \bullet & \bullet & \bullet \\
  a & b & c \\
\end{array} \quad \begin{array}{ccc}
  \bullet & \bullet & \bullet \\
  a & b & c \\
\end{array} \]

\[ \Rightarrow \quad \Rightarrow \quad \Rightarrow \]
Note: In weak 1-category (a real category) all the 2-morphisms on prev pg are identities (equals).

Recall—a weak n-category is a weak n-category where all morphisms above n are trivial.

(In a weak 1-category the 2-morphisms here are identities so \((ab)c = abc = a(bc)\).

Next consider:

\[
\begin{array}{c}
\rightarrow \rightarrow \rightarrow \rightarrow \\
a \quad b \quad c \quad d
\end{array}
\]

\[a, b, c, d \in C_i\]

These are many ways to chop this up and group together.

Corresponding to the decompositions of this into smaller cell colonies we get various ways of composing \(a, b, c, d\):

\[(ab)(cd)\]

\[((ab)c)d \quad a(b(cd))\]

\[(a(bc))d \quad a((bc)d)\]
These five came from repeated binary composition. There are more:

In tree notation:

ternary trees between 2 binary trees

all four at once

This is barycentric subdivision of a pentagon.
we have triangles

so, for each triangle, we have 6 3-morphisms
(3, i.e., their inverses)

Similarly, in a weak w-category this

cell colony

\[ \cdots a_i \cdots \]

(prev pg)

We had

\[ a, b, c, d \]

and ended

up w/a

2-diml

whose vertices correspond to binary

planar trees w/n leaves.

Higher-dimensional faces correspond
to other planar trees w/n leaves,
all

with dimensions of faces corresponding
to "number of coincidences."
James Stasheff discovered these (~1968) while working on topology.

Note: If we're in a 2-category, all those triangles commute since the 3-morphisms are all identities.

So the diagram formed by 2-morphisms in our pentagon commutes on the nose!

In traditional approach to weak 2-categories "bicategories" (Benabou 1967) there was only binary composition and associators

\[ A_{a,b,c} : (ab)c \Rightarrow a(bc) \]

satisfying "pentagon identity."

\[ (ab)(cd) \]

\[ ((ab)c)d \Rightarrow a(b(cd)) \]

\[ (a(bc))d \Rightarrow a((bc)d) \]

built up w/ associators
Examples:

1. There's a weak $w$-cat called $\text{Top}$ where objects are topological spaces, $1$-morphisms are continuous functions between them.

\[
\begin{align*}
\alpha : f &\Rightarrow g \\
x, y &\text{ top spaces} \\
f, g &\text{ cont. functs}
\end{align*}
\]

continuously defined $f \Rightarrow g$

2-morphisms $\alpha : f \Rightarrow g$ are homotopies

\[
\alpha : X \times [0,1] \rightarrow Y
\]

s.t.

\[
\begin{align*}
\alpha \big|_{X \times \{0\}} &= f \\
\alpha \big|_{X \times \{1\}} &= g
\end{align*}
\]

3-morphisms $T : \alpha \Rightarrow \beta$

are homotopies between homotopies:

\[
T : X \times [0,1]^2 \rightarrow Y
\]

That is, at each step we have a homotopy

$T(-,-,t)$ is a homotopy from $f$ to $g$, and equals $\alpha$ when $t=0$

$X \times [0,1]$

$\beta$ when $t=1$

; \ etc.
Given a weak \( w \)-category \( C \) and objects \( x, y \in \text{Ob} C \), we can define a weak \( w \)-category \( \text{ham}(x, y) \).

(If \( C \) is an \( n \)-category, \( \text{ham}(x, y) \) is an \( (n-1) \)-category.)

- Objects in \( \text{ham}(x, y) \) are \( 1 \)-morphisms \( f : x \to y \).
- Morphisms \( \alpha : f \to g \) in \( \text{ham}(x, y) \) are \( 2 \)-morphisms

\[ \alpha : f \Rightarrow g \text{ in } C, \quad f, g : x \to y \]

(Similarly, \( n \)-morphisms in \( \text{ham}(x, y) \) are \( (n+1) \)-morphisms in \( C \)).

Repeatedly applying this, given \( k \)-morphisms

\[ f, g : x \Rightarrow y \text{ in } C \] we get \( \text{ham}( f, g ) \),

which is an \( w \)-cat whose \( n \)-morphisms will be \( (n+k+1) \)-morphisms in \( C \),

these are "microcosms."
Example: Given a topological space $X$ and let $*$ be the one-point space; we get a microcosm $\text{hom}(*, X)$ which is a weak $w$-category.

Objects in $\text{hom}(*, X)$ are points.

Morphisms are paths, 2-morphisms are homotopies.

$\text{hom}(*, X)$ is called (by Grothendieck in his 600 pg letter to Quillen) the fundamental $w$-groupoid of $X$, $\pi_1(X)$.

Roughly, a weak $w$-groupoid is a weak $w$-category st every $j$-morphism $(j \geq 0)$ is an equivalence.
Recall: we have a homotopy between the id. and this path $x \Rightarrow x$

An equivalence is a $j$-morphism $f : x \Rightarrow y$

with a $j$-morphism $f^\vee : y \Rightarrow x$ which is an inverse up to equivalence (i.e., there are equivalences

$$\alpha : f f \Rightarrow 1_x$$

$$\beta : \overline{f f} \Rightarrow 1_y$$

(for $n$-categories, after $n$ they're all identities.)

This is a well-formed defn. for a weak $n$-category if we declare identities are equivalences.

In general we demand that $f$ has a "weak inverse"

(i.e.) $\exists \alpha : f f \Rightarrow 1$, $\beta : \overline{f f} \Rightarrow 1$

such that $\alpha, \beta$ have $\delta, \delta', \epsilon, \eta$

$$\delta : \alpha \overline{\alpha} \Rightarrow 1$$

$$\delta': \overline{\alpha} \alpha \Rightarrow 1$$

$$\epsilon : \beta \overline{\beta} \Rightarrow 1$$

$$\eta : \overline{\beta} \beta \Rightarrow 1$$

do this forever (the work doubles at each stage)

(next step -- $\delta, \delta', \epsilon, \eta$ each have equivalences between them, and so on)
Example: Given a topological space $X$, let $f: X \rightarrow Y$. If the homotopy class $[f]$ is trivial, then $f$ is null-homotopic.

$\{\alpha, \beta\}$ going opposite ways

$S^0$ - contractible