Structure Theorems

- A groupoid is a category with all morphisms invertible. (Here, strict = weak since we have no 2-morphisms.)

Note: morphisms must satisfy a bunch of laws; composition must be associative, left/right unit laws.

* Everything holds on the nose.

* Every morphism has an inverse.

A groupoid:

```
A = \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet
```

consists of

```
\begin{array}{c}
1_x
\end{array}
```

pieces

```
f \circ f = 1_x \Rightarrow f = f^{-1}
```

A group is a groupoid w/ one element:

```
\begin{array}{c}
\bullet
\end{array}
```

\text{elts: } 1_x, f

We need to compose } f \circ a \text{ and } ab.
$f = f^{-1}$

So, every groupoid is a "disjoint union" of connected groupoids (i.e., groupoids with 4 objects $x, y$)

If $\exists f: x \rightarrow y$ we say they're in the same connected component, i.e., an equivalence relation, since (among other things) if $f: x \rightarrow y$ then $f^{-1}: y \rightarrow x$.

Note—- symmetric property of groupoid for "disjoint union"—- no morphism going from one groupoid to other.

So— it suffices to study connected groupoids.

Note— A connected component in a groupoid is a groupoid consisting of all objects $y \iff \exists f: x \rightarrow y$ (fixing $x$) and all morphisms between them.
let \( C \) be a connected groupoid w/ set \( S \) of objects.

Pick \( x \in S \). Then \( \text{han}(x,x) \) is a group \( G_i \).

But what if we choose some other point \( y \).
Then \( \text{han}(y,y) \cong \text{han}(x,x) \) (look at picture on last pg - all pts have group \( \mathbb{Z}_2 \)).

\[ \text{han}(x,x) \cong \text{han}(y,y) \]

\( \alpha : \text{han}(x,x) \rightarrow \text{han}(y,y) \)

Choose \( f : x \rightarrow y \).

Then for \( g \in \text{han}(x,x) \)

\[ \alpha : g \rightarrow f^{-1}gf \]

Then \( \alpha \) is a group homomorphism:

\[ f^{-1}(gg')f = (f^{-1}gf)(f^{-1}g'f) \]

\( \alpha \) is 1-1 and onto since

\[ \alpha^{-1} : h \mapsto fhf^{-1} \]
So groupoid has groups in it:
all \( \text{ham}(x,x) \) endomorphisms
(morphisms from \( x \) to itself)

Claim: \( G \) and \( S \) determine \( C \) up to isomorphism,
so we'll say \( C \cong G[S] \).

\[ \text{ham}(y,y) \]
\[ f_y \]
\[ f_y^{-1} \]
\[ y \]
\[ y' \]
\[ x \]
\[ \text{pts in } S \]

\( \text{ham}(x,x) = G \)

Assuming it's connected — \( \forall y \neq x \), we can
find a morphism from \( x \) to that other pt.

\( \forall y \neq x \text{ in } S \), choose \( f_y : x \to y \).

These \( f_y \)'s give isomorphisms

\[ \alpha_y : \text{ham}(x,x) \xrightarrow{\sim} \text{ham}(y,y) \]

as we did before. We get maps from \( y \) to
itself by doing \( f_y^{-1} \) (something in \( \text{ham}(x,x) \)) \( f_y \).
If \( y = x \), choose \( \alpha_y = 1_x \) (identity)

Now note \( \forall y, y' \in C \)

\[
\alpha_{yy'} : \text{han}(y, y') \sim \rightarrow \text{han}(x, x) = G
\]

\[
h \rightarrow f_y h f_{y'}^{-1}
\]

so now we have an isomorphism \( \text{han}(x, y) \) and \( \text{han}(x, x) = G \).

What does \( \nu, h \), \( \nu', h' \), \( \nu, hh' \)

\[
o : \text{han}(y, y') \times \text{han}(y', y'') \rightarrow \text{han}(y, y'')
\]

look like if we interpret it as a map

**Quest:** \( G \times G \rightarrow G ? \)

*check this

**Answer:** \( (\alpha_{yy} h, \alpha_{yy''} h') \rightarrow (\alpha_{yy} h)(\alpha_{yy''} h') \)

\( G \times G \rightarrow G \) multiplication in \( G \)!
Structure Theorem: A group $G$ and a set $S$ determine a connected groupoid $G[S]$ with $S$ as objects and $\text{hom}(x, x) \equiv G \forall x \in G$, up to isomorphism. Canonical if we pick $x \in S$.

Any groupoid is isomorphic to a disjoint union of groupoids of this form:

$$\bigsqcup_{\alpha} G_{\alpha} [S_{\alpha}]$$

These are the connected components.

Example: Let $C$ be the groupoid whose objects are finite sets and morphisms are all 1-1 and onto functions (we want them to be invertible).

The components of $C$ are:

- 1-elt sets (all $n$-element sets are in the same connected component)
- 2-elt sets
- 3-elt sets
- etc.

Correspond to $n = 0, 1, 2, 3, \ldots$

So $S_n$ = the set of all $n$-element sets.
\[ \text{hom}(x, x) \text{ where } x \in S_n. \text{ So } x \text{ is a set of } n \text{ elements.} \]

And \( G_n = \text{the group of permutations of an } \ n\text{-element set "} n! \text{"} \)

So:

\[ C = \prod_{n=0}^{\infty} C_n \]

our name for the perm. group \( S_n \)

**Defn:** We call a groupoid **skeletal** if all components have only one object.

(just a bunch of groups)

If \( C \) is **skeletal**, \( C = \prod_{\alpha} G_{\alpha} \)

If in prev. example, we make it skeletal, we have just one 1-elt set, one 2-elt set, etc...
2-Groupoids

Now we'll just study strict 2-groupoids, i.e., strict 2-categories with all morphisms and invertible "on the nose". (Later we'll do weak ones.)

This 2-morphism isn't going from inverse to inverse.

\[ \alpha : f \Rightarrow g \]
\[ \text{but } \alpha : f'' \Rightarrow g^{-1} \]

If we forget about 2-morphisms, we have a groupoid.

Given a 2-groupoid \( C \), we could ignore the 2-morphisms and get a groupoid \( \tilde{C} \).

\( \text{hom}(x,x) \) has morphisms and 2-morphisms.

Given \( x \in C \), the set of morphisms \( f : x \to x \) is a group.

Also, the set of 2-morphisms \( \alpha : 1_x \Rightarrow 1_x \) forms a group, under vertical composition.

\[ \alpha \beta : 1_x \Rightarrow 1_x \]
This 2nd group, \( \text{hom}(1_x, 1_x) \) is interesting because it is abelian.

Notice - this group also has another product given by horizontal composition:

\[
\alpha \cdot \beta : 1_x \Rightarrow 1_x
\]

We also have:

\[
(\alpha \cdot \beta)(\beta \cdot \delta) = (\alpha \beta) \cdot (\beta \delta)
\]

Vertical composition makes this into a group but we don’t know that horizontal comp. does. (in fact it does)

* Claim: \( 1_{1_x} \) is also identity for horiz. comp.

\[
\alpha \cdot 1_{1_x} = \alpha \quad \text{and} \quad 1_{1_x} \cdot \alpha = \alpha
\]

\[
= (\alpha \cdot 1_{1_x})(1_{1_x} \cdot 1_{1_x})
\]
So -

\[(\alpha \cdot 1_{1_x}) = (\alpha \cdot 1_{1_x})(1_{1_x} \cdot 1_{1_x})\]

\[\Rightarrow 1_{1_x} \cdot 1_{1_x} = 1_{1_x} \text{ (identity)}\]

\[\hom(x, x) \text{ consists of 1-morphisms } f : x \to x\]

and all 2-morphisms between them.