

Bernoulli #'s

1.) $e^{t\partial} = \sum_{k \geq 0} \frac{t^k}{k!} \frac{d^k}{dz^k}$. Since any $f \in \mathcal{E}$ can be written as a power series that converges everywhere, we just need to work out what $e^{t\partial}$ does to monomials.

$$\begin{aligned} e^{t\partial} z^n &= \sum_{k \geq 0} \frac{t^k}{k!} \frac{d^k z^n}{dz^k} \\ &= \sum_{k=0}^n \frac{t^k}{k!} n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) z^{n-k} \\ &= \sum_{k=0}^n \frac{t^k}{k!} \frac{n!}{(n-k)!} z^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} z^{n-k} t^k \\ &= (z+t)^n \end{aligned}$$

Using the obvious linearity of $e^{t\partial}$, we therefore get, $\forall f \in \mathcal{E}$,

$$(e^{t\partial} f)(z) = e^{t\partial} \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} (z+t)^n = f(z+t).$$

2) $\Delta: \mathcal{E} \rightarrow \mathcal{E}$ is defined by $(\Delta f)(z) := f(z+1) - f(z)$.

For any entire function f we have, $\forall z \in \mathbb{C}$,

$$((e^{\partial} + 1)f)(z) = (e^{\partial} f)(z) + f(z) = f(z+1) - f(z)$$

So: $e^{\partial} + 1 = \Delta$

3.) Let $F \in \mathcal{E}$ and $f = \Delta F$. Then:

$$\begin{aligned} \sum_{k=0}^{n-1} f(k) &= \sum_{k=0}^{n-1} \Delta F(k) \\ &= \sum_{k=0}^{n-1} [F(k+1) - F(k)] && \text{(telescoping series)} \\ &= F(n) - F(0) \end{aligned}$$

(Since we might not always want "1" to be the step size for our discretization, we could define more generally

$$\Delta_h = \frac{e^{ha} - 1}{h} \quad \left(\text{so that } \Delta_h f = \frac{f(x+h) - f(x)}{h} \right) \quad \checkmark$$

Then the discrete version of the FTC says: If $F \in \mathcal{E}$ and $\Delta_h F = f$, then

$$\sum_{k=0}^{n-1} h f(kh) = F(nh) - F(0).$$

$$\begin{aligned} \text{Pf: } \sum_{k=0}^{n-1} h f(kh) &= \sum_{k=0}^{n-1} h \Delta_h F(kh) = \sum_{k=0}^{n-1} F(kh+h) - F(kh) \\ &= \sum_{k=0}^{n-1} F((k+1)h) - F(kh) \\ &= F(nh) - F(0) \end{aligned}$$

4.) Since $\int_\gamma f$ depends only on the endpoints of the path γ when f is entire, we may speak unambiguously of $\int_0^z f(u) du$. Also, the Taylor series of an analytic function may be integrated term-by-term, so we only need to show $\partial \bar{\partial}^{-1} z^n = z^n \quad \forall n \in \mathbb{N}$. This is straightforward:

$$\partial \bar{\partial}^{-1} z^n = \frac{d}{dz} \int_0^z u^n du = \frac{d}{dz} \frac{z^{n+1}}{n+1} = z^n. \quad \checkmark$$

To see that $a^{-1}af$ is not necessarily equal to f , we need only consider the entire function 1:

$$a^{-1}a1 = \int_0^z \left(\frac{d}{du} 1 \right) du = \int_0^z 0 du = 0$$

(If we wanted to be silly we could define an equivalence relation \sim , decreeing entire functions to be equivalent if they differ by a constant. Then we could define $a: \mathcal{E}/\sim \rightarrow \mathcal{E}$ and $a^{-1}: \mathcal{E} \rightarrow \mathcal{E}/\sim$ in the obvious way, forcing a and a^{-1} to be bona fide mutual inverses). \checkmark

$$5.) \quad \frac{x}{e^x - 1} = \sum_{k \geq 0} B_k \frac{x^k}{k!}$$

$$\begin{aligned} x &= (e^x - 1) \sum_{k \geq 0} B_k \frac{x^k}{k!} \\ &= \left(\sum_{j \geq 0} \frac{x^j}{j!} - 1 \right) \sum_{k \geq 0} B_k \frac{x^k}{k!} \\ &= \sum_{j \geq 1} \frac{x^j}{j!} \sum_{k \geq 0} B_k \frac{x^k}{k!}. \end{aligned}$$

If we think of x as the operator "multiplication by x " (i.e. the creation operator a^+), then we have

$$a^+ = \sum_{j \geq 1} \frac{(a^+)^j}{j!} \sum_{k \geq 0} B_k \frac{(a^+)^k}{k!}$$

or, dualizing:

$$a = \sum_{j \geq 1} \frac{a^j}{j!} \sum_{k \geq 0} B_k \frac{a^k}{k!}$$

$$= \Delta \frac{a}{e^a - 1}.$$

nice!

$$6.) \Delta \Delta^{-1} f = \Delta \left(\frac{a}{e^a - 1} a^{-1} \right) f$$

definition of Δ^{-1}

$$= \left(\Delta \frac{a}{e^a - 1} \right) a^{-1} f$$

associativity of operators

$$= a a^{-1} f$$

result of part 5: $a = \Delta \frac{a}{e^a - 1}$

$$= 1 f$$

a^{-1} is a right inverse of a (part 4)

$$= f$$

But, suppose f is the constant function 1. Then

$$\Delta^{-1} \Delta f(z) = \Delta^{-1} (f(\cdot + 1) - f(\cdot))(z)$$

$$= \Delta^{-1}(0)$$

the zero function $z \mapsto 0$

$$= \frac{a}{e^a - 1} a^{-1}(0)$$

$$= \frac{a}{e^a - 1} \int_0^z 0 du$$

$$= \frac{a}{e^a - 1} (0)$$

$$= \sum_{k \geq 0} B_k \frac{a^k}{k!} (0)$$

$$= 0 \neq 1 \quad \text{so} \quad \Delta^{-1} \Delta (z \mapsto 1) \neq (z \mapsto 0)$$

$$\neq (z \mapsto 1).$$



7.) The result $\frac{B_0}{1!0!} + \frac{B_1}{(i-1)!1!} + \dots + \frac{B_i}{1!(i-1)!}$ is most easily seen if we write out the terms in the expansion of x (from part 5) as an array, as follows:

$$\begin{aligned}
 x &= \sum_{j=1}^{\infty} \frac{x^j}{j!} \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k \\
 &= \left(\frac{x}{1!} \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k \right) + \left(\frac{x^2}{2!} \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k \right) + \left(\frac{x^3}{3!} \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k \right) + \dots \\
 &= \left(\frac{B_0}{1!0!} x^{1+0} + \frac{B_1}{1!1!} x^{1+1} + \frac{B_2}{1!2!} x^{1+2} + \frac{B_3}{1!3!} x^{1+3} + \dots \right) \\
 &\quad + \left(\frac{B_0}{2!0!} x^{2+0} + \frac{B_1}{2!1!} x^{2+1} + \frac{B_2}{2!2!} x^{2+2} + \frac{B_3}{2!3!} x^{2+3} + \dots \right) \\
 &\quad + \left(\frac{B_0}{3!0!} x^{3+0} + \frac{B_1}{3!1!} x^{3+1} + \frac{B_2}{3!2!} x^{3+2} + \frac{B_3}{3!3!} x^{3+3} + \dots \right) \\
 &\quad + \left(\frac{B_0}{4!0!} x^{4+0} + \frac{B_1}{4!1!} x^{4+1} + \frac{B_2}{4!2!} x^{4+2} + \dots \right) \\
 &\quad + \left(\frac{B_0}{5!0!} x^{5+0} + \frac{B_1}{5!1!} x^{5+1} + \dots \right) \\
 &\quad + \dots \\
 &\quad \vdots
 \end{aligned}$$

LINEAR TERMS
 QUADRATIC TERMS
 CUBIC TERMS
 QUARTIC TERMS
 QUINTIC TERMS

Since on the LHS we have just x , equating coefficients gives:

$$\begin{aligned}
 1 &= \frac{B_0}{1!0!} \\
 0 &= \frac{B_0}{1!0!} + \frac{B_1}{1!1!} \\
 &\vdots \\
 0 &= \frac{B_0}{i!0!} + \frac{B_1}{(i-1)!1!} + \dots + \frac{B_i}{1!(i-1)!} \\
 &\vdots
 \end{aligned}$$



8.) So far we have:

$$1 = \frac{B_0}{1!0!} \quad i=1$$

$$0 = \frac{B_0}{2!0!} + \frac{B_1}{1!1!} \quad i=2$$

$$0 = \frac{B_0}{3!0!} + \frac{B_1}{2!1!} + \frac{B_2}{1!2!} \quad i=3$$

$$0 = \frac{B_0}{4!0!} + \frac{B_1}{3!1!} + \frac{B_2}{2!2!} + \frac{B_3}{1!3!} \quad i=4$$

and so on. But if we multiply the i th row by $i!$, we get

$$1 = \binom{1}{1} B_0$$

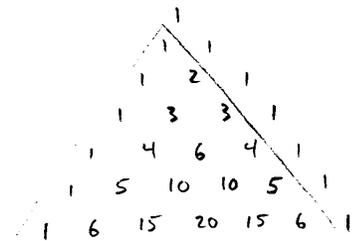
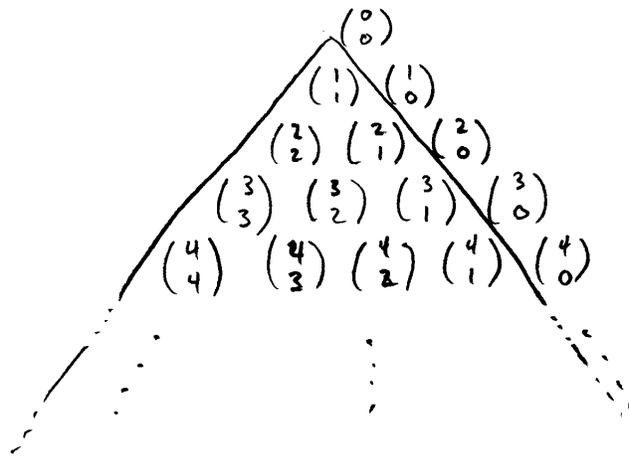
$$\binom{r}{s} = \frac{r!}{s!(r-s)!}$$

$$0 = \binom{2}{2} B_0 + \binom{2}{1} B_1$$

$$0 = \binom{3}{3} B_0 + \binom{3}{2} B_1 + \binom{3}{1} B_2$$

$$0 = \binom{4}{4} B_0 + \binom{4}{3} B_1 + \binom{4}{2} B_2 + \binom{4}{1} B_3$$

and so on, where the coefficients are obviously the indicated portion of Pascal's triangle:



9.) I already showed that $B_0 = 1$.

$$\bullet \quad 2B_1 = -B_0 = -1 \implies B_1 = -\frac{1}{2}$$

$$\bullet \quad 3B_2 = -(3B_1 + B_0) \\ = -\left(-\frac{3}{2} + 1\right) = \frac{1}{2} \implies B_2 = \frac{1}{6}$$

$$\bullet \quad 4B_3 = -(6B_2 + 4B_1 + B_0) \\ = -(1 - 2 + 1) \\ = 0 \implies B_3 = 0$$

$$\bullet \quad 5B_4 = -(10B_3 + 10B_2 + 5B_1 + B_0) \\ = -(0 + \frac{5}{3} + -\frac{5}{2} + 1) \\ = -\frac{1}{6} \implies B_4 = -\frac{1}{30}$$

~~To find an explicit formula for $\sum i^4$, we'll need one more:~~

$$\bullet \quad 6B_5 = -(15B_4 + 20B_3 + 15B_2 + 6B_1 + B_0) \\ = -\left(-\frac{1}{2} + 0 + \frac{5}{2} + (-3) + 1\right) = 0 \implies B_5 = 0$$

↖ only needed this part
for the wrong formula for $\sum i^4$
see next page.

$$10.) \quad \sum_{i=1}^n i^4 = \frac{1}{5} \sum_{k=0}^4 B_k \binom{5}{k} (n+1)^{5-k} \\ = \frac{1}{5} \left[(n+1)^5 - \frac{1}{2} \cdot 5(n+1)^4 + \frac{1}{6} \cdot 10(n+1)^3 - \frac{1}{30} \cdot 5(n+1) \right] \\ = \left[\frac{(n+1)^4}{5} - \frac{(n+1)^3}{2} + \frac{(n+1)^2}{3} - \frac{1}{30} \right] (n+1)$$

(I checked this for a couple of low order examples, and indeed it gives 979 for the sum of the first five integers to the fourth power, and 2275 for the first six. That's enough to convince me I didn't screw up any of the algebra!)

$$\begin{aligned}
 11) \quad \frac{(B+n+1)^{P+1} - B^{P+1}}{P+1} &= \frac{1}{P+1} \left[\sum_{k=0}^{P+1} \binom{P+1}{k} B^k (n+1)^{P+1-k} - B^{P+1} \right] \quad (\text{binomial thm}) \\
 &= \frac{1}{P+1} \sum_{k=0}^P B^k \binom{P+1}{k} (n+1)^{P+1-k} \\
 &= \frac{1}{P+1} \sum_{k=0}^P B_k \binom{P+1}{k} (n+1)^{P+1-k} \quad B^k \mapsto B_k
 \end{aligned}$$

which is not the same as the formula on p. 4 of the homework, which says

$$\sum_{i=1}^n i^P = \frac{1}{P+1} \sum_{k=1}^{P+1} B_k \binom{P+1}{k} (n+1)^{P+1-k} \quad (\text{wrong!})$$

I've fixed this on the web version -

It appears that this one is the one that's wrong. thanks!

The correct formula is

JB

$$\sum_{i=1}^n i^P = \frac{1}{P+1} \sum_{k=0}^P B_k \binom{P+1}{k} (n+1)^{P+1-k}$$

What happens when $n=1$?

12.) I'm afraid I'm out of time to try categorifying. :-)