QUANTUM GRAVITY HOMEWORK 6

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1. For $e^{ta} : \mathcal{E} \to \mathcal{E}$ defined by

$$e^{ta} = \sum_{k \geq 0} \frac{(ta)^k}{k!},$$

and $f(z) = \sum_{j \geq 0} c_j z^j$, we have

$$
\begin{aligned}
e^{ta} f(z) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(ta)^k}{k!} c_n z^n \\
&= \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \frac{t^k}{k!} n(n-1)(n-2) \cdots (n-k+2)(n-k+1) z^{n-k} \\
&= \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^{n-k} t^k \\
&= \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \binom{n}{k} z^{n-k} t^k \\
&= \sum_{n=0}^{\infty} c_n (z+t)^n \\
&= f(z+t)
\end{aligned}
$$

2. Using 1, we have

$$(e^a - 1) f(z) = (e^a f - f)(z) = e^a f(z) - f(z) = f(z+1) - f(z) = \Delta f(z).$$

3. Suppose $F \in \mathcal{E}$ and $\Delta F = f$. Then

$$
\begin{aligned}
F(n) - F(0) &= \underbrace{F(n) - F(n-1)}_{f(n-1)} + \underbrace{F(n-1) - F(n-2)}_{f(n-2)} + \ldots \\
&\qquad \underbrace{\ldots - F(2)}_{f(2)} + \underbrace{F(2) - F(1)}_{f(1)} + \underbrace{F(1) - F(0)}_{f(0)} \\
&= f(n-1) + f(n-2) + \ldots + f(2) + f(1) + f(0) \\
&= \sum_{j=0}^{n-1} f(j)
\end{aligned}
$$

4. We define $a^{-1} : \mathcal{E} \to \mathcal{E}$ by

$$(a^{-1})f(z) = \int_0^z f(u)du.$$

Then

$$(aa^{-1})f(z) = a(a^{-1}f)(z) = a(\int_0^z f(u)du) = \tfrac{d}{dz}\int_0^z f(u)du = f(z),$$

by the FTOC. However, suppose $f(z) = 1$, $\forall z$. Then $f \in \mathcal{E}$ but

$$a^{-1}af(z) = a^{-1}(\tfrac{d}{dz}1) = a^{-1}(0) = \int_0^z 0\, du = 0 \neq 1 = f(z).$$

5. Since we have

$$\frac{x}{e^x - 1} = \sum_{k \geq 0} B_k \frac{x^k}{k!},$$

we can multiply both sides by $e^x - 1$ to get

$$x = (e^x - 1) \sum_{k \geq 0} B_k \frac{x^k}{k!}.$$

Then

$$e^x = \sum_{j \geq 0} \tfrac{x^j}{j!} \implies e^x - 1 = \sum_{j \geq 1} \tfrac{x^j}{j!},$$

and we can write

$$x = \left(\sum_{j \geq 1} \frac{x^j}{j!} \right) \left(\sum_{k \geq 0} B_k \frac{x^k}{k!} \right).$$

This term on the right is a power series. It looks like two, but after multiplying them as a Cauchy product, it is clearly just one power series (see 7). Since we may always define an operator by a power series, this allows us to write

$$a = \left(\sum_{j \geq 1} \frac{a^j}{j!} \right) \left(\sum_{k \geq 0} B_k \frac{a^k}{k!} \right). \tag{1}$$

Now since

$$\sum_{j \geq 1} \frac{a^j}{j!} = e^{ta} - 1 = \Delta$$

by 2, and

$$\sum_{k \geq 0} B_k \frac{a^k}{k!} = \frac{a}{e^a - 1}$$

by definition, we can rewrite (1) as

$$a = \Delta \frac{a}{e^a - 1}.$$

6. Using $\Delta^{-1} = \frac{a}{e^a - 1} a^{-1}$, we have

$$\Delta\Delta^{-1}f = \Delta\left(\frac{a}{e^a - 1} a^{-1}\right) f \qquad\qquad \text{def of } \Delta^{-1}$$

$$= \left(\Delta \frac{a}{e^a - 1}\right) a^{-1}f \qquad\qquad \text{operator associativity}$$

$$= aa^{-1}f \qquad\qquad \text{by 5}$$

$$= f \qquad\qquad \text{by 4}$$

However, we don't necessarily have $\Delta^{-1}\Delta f = f$. For example, suppose $f(z) = e^{2\pi i z}$. Then $f \in \mathcal{E}$, but

$$\Delta^{-1}\Delta f = \Delta^{-1}(\Delta f)$$

$$= \Delta^{-1}\left(e^{2\pi i(z+1)} - e^{2\pi i z}\right)$$

$$= \Delta^{-1}\left(e^{2\pi i z}e^{2\pi i} - e^{2\pi i z}\right)$$

$$= \Delta^{-1}\left(e^{2\pi i z} - e^{2\pi i z}\right)$$

$$= \Delta^{-1}(0)$$

$$= 0$$

$$\neq e^{2\pi i z} = f(z).$$

7. From 5 we have

$$x = \sum_{j\geq 1} \frac{x^j}{j!} \sum_{k\geq 0} B_k \frac{x^k}{k!}$$

$$= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(B_0\frac{1}{0!} + B_1\frac{x^1}{1!} + B_2\frac{x^2}{2!} + B_3\frac{x^3}{3!} + \dots\right)$$

$$= B_0 x + B_1\frac{x^1 x^1}{1!1!} + B_2\frac{x^1 x^2}{1!2!} + B_3\frac{x^1 x^3}{1!3!} + B_4\frac{x^5}{1!4!} + B_5\frac{x^6}{1!5!} + \dots$$

$$+ B_0\frac{x^2}{2!0!} + B_1\frac{x^2 x^1}{2!1!} + B_2\frac{x^2 x^2}{2!2!} + B_3\frac{x^5}{2!3!} + B_4\frac{x^6}{2!4!} + \dots$$

$$+ B_0\frac{x^3}{3!0!} + B_1\frac{x^3 x^1}{3!1!} + B_2\frac{x^5}{3!2!} + B_3\frac{x^6}{3!3!} + \dots$$

$$+ B_0\frac{x^4}{4!0!} + B_1\frac{x^5}{4!1!} + B_2\frac{x^6}{4!2!} + \dots$$

$$= \sum_{n\geq 1}\sum_{k=0}^{n-1} B_k \frac{x^k}{k!}\frac{x^{n-k}}{(n-k)!}$$

$$= \sum_{n\geq 1}\sum_{k=0}^{n-1} B_k \frac{x^n}{n!}\frac{n!}{k!(n-k)!}$$

$$x = \sum_{n \geq 1} \frac{x^n}{n!} \sum_{k=0}^{n-1} \binom{n}{k} B_k \tag{2}$$

Then taking the coefficients of x^{i+1} for $i \geq 1$,

$$
\begin{aligned}
0 &= \frac{1}{(i+1)!} \sum_{k=0}^{i} \binom{i+1}{k} B_k \\
&= \frac{1}{(i+1)!} \left(B_0 \frac{(i+1)!}{0!(i+1)!} + B_1 \frac{(i+1)!}{1!(i)!} + + \ldots + B_i \frac{(i+1)!}{(i)!1!} \right) \\
&= \frac{B_0}{0!(i+1)!} + \frac{B_1}{1!(i)!} + \frac{B_2}{2!(i-1)!} + \ldots + \frac{B_i}{(i)!1!}.
\end{aligned}
$$

8. From (2) above, we can collect coefficients for all terms and immediately obtain

$$
\begin{aligned}
1 &= 1B_0 \\
0 &= 1B_0 + 2B_1 \\
0 &= 1B_0 + 3B_1 + 3B_2 \\
0 &= 1B_0 + 4B_1 + 6B_2 + 4B_3 \\
0 &= 1B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 \\
0 &= 1B_0 + 6B_1 + 15B_2 + 20B_3 + 15B_4 + 6B_5
\end{aligned}
$$

9. From the first coefficient comparison above, we see $B_0 = 1$. Plugging this into the equations from 8, we obtain:

$$B_0 = 1 \qquad B_1 = -\frac{1}{2} \qquad B_2 = \frac{1}{6} \qquad B_3 = 0 \qquad B_4 = -\frac{1}{30} \qquad B_5 = 0$$

10. Using the previously obtained formula

$$\sum_{i=1}^{n} i^p = \frac{1}{p+1} \sum_{k=0}^{p+1} B_k \binom{p+1}{k} (n+1)^{p+1-k}$$

with $p = 4$, we obtain

$$1^4 + 2^4 + \cdots + n^4 = \frac{1}{5} \sum_{k=0}^{5} B_k \binom{5}{k} (n+1)^{5-k}$$

$$= \frac{1}{5} \left(1B_0(n+1)^5 + 5B_1(n+1)^4 + 10B_2(n+1)^3 + 10B_3(n+1)^2 + 5B_4(n+1)^1 + B_5 \right)$$

$$= \frac{1}{5} B_0(n+1)^5 + B_1(n+1)^4 + 2B_2(n+1)^3 + 2B_3(n+1)^2 + B_4(n+1)^1 + \frac{1}{5} B_5$$

$$= \frac{1}{5}(n+1)^5 - \frac{1}{2}(n+1)^4 + \frac{1}{3}(n+1)^3 - \frac{1}{30}(n+1)$$

11. The formula

$$1^p + 2^p + \cdots + n^p = \frac{(B+n+1)^{p+1} - B^{p+1}}{p+1}$$

is just a nifty way of writing

$$\sum_{i=1}^{n} i^p = \frac{1}{p+1} \sum_{k=0}^{p+1} B_k \binom{p+1}{k} (n+1)^{p+1-k}$$

When $n = 0$ it reduces to

$$(B+1)^{p+1} = B^{p+1}$$

which is just a nifty way of writing the recursive formula for Bernoulli numbers in terms of Pascal's triangle, which we have already seen in problem 8.