“Bernoulli” Numbers

Jeff “Morton”

1. We're interested in the operator $\mathrm{e}^{ta} = \sum_{k\geqslant 0} \frac{(ta)^k}{k!}$, which is to say $\sum_{k\geqslant 0} \frac{t^k}{k!} \cdot \frac{\mathrm{d}^k}{\mathrm{d}z^k}$. Applying this to some function $f(z) \in \mathcal{E}$ to get $(\mathrm{e}^{ta} f)(z) = \sum_{k\geqslant 0} \frac{t^k}{k!} \cdot \frac{\mathrm{d}^k f(z)}{\mathrm{d}z^k} = f(z) + \sum_{k\geqslant 1} \frac{t^k}{k!} \cdot \frac{\mathrm{d}^k f(z)}{\mathrm{d}z^k}$, we note that since f is entire, Taylor's theorem says that this expansion as a power series in t about z converges to the value of $f(z+t)$. This being so for every $z \in \mathbb{C}$, we have that $(\mathrm{e}^{ta} f)(z) = f(z+t)$.
2. The difference operator Δ is defined as $(\Delta f)(z) = f(z+1) - f(z) = (\mathrm{e}^{1a} f)(z) - f(z)$, for all f, z. So as an operator, $\Delta = \mathrm{e}^{1a} - 1 = \mathrm{e}^{a} - 1$.
3. If $\Delta F = f$, we have $\sum_{i=0}^{n-1} f(i) = \sum_{i=0}^{n-1} (\Delta F)(i) = \sum_{i=0}^{n-1} F(i+1) - F(i)$. This is a telescoping sum, in which every value in the sum except the first and last appears twice, with opposite signs (first positive, then negative with the next value of of the summation index). Cancellation leaves the $F(i+1)$ term for the case $i = n-1$ and the $F(i)$ term for $i = 0$. Thus, we have $\sum_{i=0}^{n-1} f(i) = F(n) - F(0)$.
4. Given any entire function $f \in \mathcal{E}$, we have

   $$(aa^{-1}f) = \frac{\mathrm{d}}{\mathrm{d}z}(a^{-1}f)(z) = \left(\frac{\mathrm{d}}{\mathrm{d}z'} \int_0^{z'} f(u)\mathrm{d}u\right)(z).$$

   As a complex-valued function on the real line, the Fundamental Theorem of Calculus means that the derivative of $\int_0^{z'} f(u)\mathrm{d}u$ with respect to z' at z is just the value of f there, $f(z)$. Since f is entire as a function $f : \mathbb{C} \to \mathbb{C}$, the complex derivative exists everywhere and is just the same as the derivative as a function $f' : \mathbb{R} \to \mathbb{C}$. So in fact we have for all z that $(aa^{-1}f)(z) = f(z)$, so $aa^{-1}f = f$ (and this is true $\forall f \in \mathcal{E}$, hence in fact as an operator $aa^{-1} = \mathrm{Id}_{\mathcal{E}}$). On the other hand, since the integral of a function is only defined up to a constant, a is not a *right* inverse of a^{-1}: if $f \equiv k$ for any constant $k \neq 0$, we have:

   $$(a^{-1}af)(z) = \int_0^z (af)(u)\mathrm{d}u = \int_0^z \left(\frac{\mathrm{d}}{\mathrm{d}u}k\right)\mathrm{d}u = \int_0^z 0\mathrm{d}u = 0$$

   This is, in general, not $f(z)$, which is k for all z. Thus, a is a left inverse only for a^{-1}, so called. In categorical terms, both maps are endomorphisms of $\mathcal{E}$. Seen this way, a is an epimorphism (indeed, a split epi since it has a right inverse - and in fact is surjective as a set-map since every function in $\mathcal{E}$ is the derivative of something in $\mathcal{E}$) and likewise a^{-1} is a monomorphism (indeed, a spilt mono since it has a left inverse - and in fact is injective as a set map, since there is exactly one function in $\mathcal{E}$ whose integral any given f is, namely its derivative).
5. The Bernoulli numbers are the coefficients B_k in the expression $\frac{z}{\mathrm{e}^z - 1} = \sum_{k\geqslant 0} B_k \frac{z^k}{k!}$, which is an entire function. The function $e^z - 1$ is also entire

and only zero at $z = 0$, where it has a zero of first order (hence $\frac{z}{\mathrm{e}^z - 1}$ being entire). The power series for this function is $\mathrm{e}^z - 1 = \sum_{j \geqslant 0} \frac{z^j}{j!} - 1 = \sum_{j \geqslant 1} \frac{z^j}{j!}$. Since both of these are entire functions, the power series converge everywhere in $\mathbb{C}$, and the product is also entire, so we can write the product of the two functions (which is just z, of course) as the product of the two power series:

$$z = \sum_{j \geqslant 1} \frac{z^j}{j!} \sum_{k \geqslant 0} B_k \frac{z^k}{k!}$$

Now, to see that the same applies when we replace the complex variable z by the differential operator $a = \frac{\mathrm{d}}{\mathrm{d}z}$, we can note that the Fourier transform of this differential operator is multiplication by z (i.e. if we take functions to their Fourier transforms, the action of a is to take the transform of a function f, say $\bar{f}$, to $z\bar{f}$, and a^k acting on f takes $\bar{f}$ to $z^k \bar{f}$). Thus, by the above and by linearity of the Fourier transform, the effect of a on $\bar{f}$ is:

$$a = \sum_{j \geqslant 1} \frac{a^j}{j!} \sum_{k \geqslant 0} B_k \frac{a^k}{k!}$$

Now, by definition of Δ and $\frac{a}{\mathrm{e}^a - 1}$, this just says that $a = \Delta \frac{a}{\mathrm{e}^a - 1}$.

6. To see that Δ^{-1} is a right-inverse of Δ, note that for any $f \in \mathcal{E}$, we have

$$\begin{array}{rcll}
(\Delta\Delta^{-1} f)(z) & = & \left(\Delta \frac{a}{\mathrm{e}^a - 1} a^{-1} f\right)(z) & (\text{definition of } \Delta^{-1}) \\
& = & \left(\Delta \frac{a}{\mathrm{e}^a - 1} \left(\int_0^z f(u)\mathrm{d}u\right)\right) & (\text{definition of } a^{-1}) \\
& = & \left(a \left(\int_0^z f(u)\mathrm{d}u\right)\right) & (\text{by part 5}) \\
& = & \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z f(u)\mathrm{d}u & \\
& = & f(z) &
\end{array}$$

So in fact $\Delta\Delta^{-1} f = f$. On the other hand, the converse need not be so:

$$\begin{array}{rcl}
(\Delta^{-1}\Delta f)(z) & = & \left(\frac{a}{\mathrm{e}^a - 1} a^{-1} \Delta f\right)(z) \\
& = & \left(\frac{a}{\mathrm{e}^a - 1} a^{-1}((\mathrm{e}^a - 1)f)\right) \\
& = & \left(\frac{a}{\mathrm{e}^a - 1} \int_0^z \left(\sum_{j \geqslant 1} \frac{a^j}{j!}\right)(f)\mathrm{d}u\right) \\
& = & \left(\sum_{k \geqslant 0} B_k \frac{a^k}{k!} \int_0^z \left(\sum_{j \geqslant 1} \frac{a^j}{j!}(f)\right)\mathrm{d}u\right) \\
& = & \left(\sum_{k \geqslant 0} B_k \frac{a^k}{k!} \left(\sum_{j \geqslant 1} \int_0^z \left(\frac{a^j}{j!} f\right)\mathrm{d}u\right)\right)
\end{array}$$

This last step makes sense by linearity and since f is entire, so every derivative exists everywhere: the sum converges since $(\mathrm{e}^a - 1)f$ is also entire.

Now, since the integral is only a right-inverse of a, this will not necessarily be the same as f. If the integral were a left-inverse of a, we could pass the integral through the derivatives in front of f and get back f by part 5. However, this is not guaranteed to work, and we make get a constant of integration. Thus, $(\Delta^{-1}\Delta f)(z)$ may not be equal to $f(z)$.

7. We had seen that $z = \sum_{j \geqslant 1} \frac{z^j}{j!} \sum_{k \geqslant 0} B_k \frac{z^k}{k!}$ and if we equate coefficients of powers of z, we find that every coefficient of the right hand side is 0 except

for the coefficient of z^1, which is 1. Now, the coefficient of z^i on the right hand side will be

$$\sum_{\substack{j+k=i \\ j>0}} \frac{1}{j!} \cdot \frac{B_k}{k!} = \sum_{j=1}^{i} \frac{1}{j!} \cdot \frac{B_{i-j}}{(i-j)!}$$

If we expand this sum for any $i \neq 1$, so that the whole sum is 0, we find:

$$0 = \frac{1}{1!} \cdot \frac{B_{i-1}}{(i-1)!} + \ldots + \frac{1}{(i-1)!} \cdot \frac{B_1}{1!} + \frac{1}{i!} \cdot \frac{B_0}{0!}$$

8. The relations we found in part 7 give expressions in the B_j which sum to 0, one for each value of i greater than 1. In each case, we have fractional coefficients which can be cleared by multiplying the whole expression on the right hand side by $i!$, in which case we get the relations:

$$0 = \sum_{\substack{j+k=i \\ j>0}} \frac{i!}{j!k!} B_{i-j} = \sum_{\substack{j+k=i \\ j>0}} \binom{i}{j} B_{i-j}$$

Notice that the coefficients of the B_j are the same as the binomial coefficients from Pascal's triangle, as we had hoped.

9. To find out B_0, recall that we defined the B_k to be the coefficients in z of the power series for the function $\frac{z}{e^z-1}$ extended to equal 1 at $x = 0$. B_0 is the constant coefficient for the power series about 0, and is therefore 1. Using the relations from part 8, this implies that:
$0 = 1(1) + 2(B_1)$, hence $B_1 = -\frac{1}{2}$
$0 = 1(1) + 3(-\frac{1}{2}) + 3(B_2) = -\frac{1}{2} + 3B_2$, hence $B_2 = \frac{1}{6}$
$0 = 1(1) + 4(-\frac{1}{2}) + 6(\frac{1}{6}) + 4(B_3) = 0 + 4(B_3)$, hence $B_3 = 0$
$0 = 1(1) + 5(-\frac{1}{2}) + 10(\frac{1}{6}) + 10(0) + 5(B_4) = \frac{1}{6} + 5B_4$, hence $B_4 = -\frac{1}{30}$.
$0 = 1(1) + 6(-\frac{1}{2}) + 15(\frac{1}{6}) + 20(0) + 15(-\frac{1}{30}) + 6(B_5)$, hence $B_5 = 0$

10. We had seen that

$$\textstyle\sum_{i=1}^{n} i^p = \frac{1}{p+1} \sum_{k=0}^{p} B_k \binom{p+1}{k} (n+1)^{p+1-k}$$

Applying this to the situation where $p = 4$, we find

$$\begin{aligned} \textstyle\sum_{i=1}^{n} i^4 &= \textstyle\frac{1}{5} \sum_{k=0}^{4} B_k \binom{5}{k} (n+1)^{5-k} \\ &= \textstyle\frac{1}{5} \left[(1)(1)(n+1)^5 + \left(-\frac{1}{2}\right)(5)(n+1)^4 + \left(\frac{1}{6}\right)(10)(n+1)^3 + \left(-\frac{1}{30}\right)(5)(n+1)\right] \\ &= \textstyle\frac{(n+1)^5}{5} - \frac{(n+1)^4}{2} + \frac{(n+1)^3}{3} - \frac{(n+1)}{30} \end{aligned}$$

11. The binomial expansion for $(B + (n+1))^{p+1}$ is $\sum_{k=0}^{p+1} \binom{p+1}{k} B^k (n+1)^{p+1-k}$. Identifying B^k with B_k and dividing by $p + 1$ gives the expression above.

12. The first and most obvious reason it's difficult to categorify this business is the presence of negative coefficients, which means we can't categorify using

ordinary species. (Cubical species could help here, though they are not necessary at first.) Another problem is that A^{-1} and Δ^{-1} will not be associated with natural transformations between structure types, since they necessarily involves an arbitrary choice (of elements to remove, in each case). Here follow some comments on categorifying the results from various parts of this computation:

1. If we define E^{tA} to be the operation on structure types $E^{TA} = \sum_{k \geqslant 0} \frac{(tA)^k}{k!}$, this amounts to an operation taking a structure type and producing the "sum" (union) of types which apply (TA) some number k of times, over all k. For each particular case k, in the case $t = 1$, this is simply taking the "derivative" k times, which gives a new structure whose effect on a set S is to put the original structure on the set $S + k$ - the $k!$ denominator reduces be the action of a permutation group, meaning these elements are unordered. When T is general, we interpret this as meaning that the elements we add are T-coloured.
   The result here is that $(E^{TA}F)(Z) \cong F(Z + T)$ - that is, putting the structure $E^{tA}F$ on a set is the same as putting an F structure on a set of things which are either one-element sets or members of T, the set of colours we could paint the new elements we add in the definition of E^{TA}. That is, we think of these not as elements of a set contributing to the cardinality of the set S on which we put the E^{TA}-structure, but as "just colours". (Here we're using the interpretation of composition that a $F(Z + T)$ structure is an F structure on sets of $Z + T$ structures, i.e. things which are either a one-element set or a colour from T.

2. The Δ operator, applied to a structure type F, should satisfy $\Delta F(Z) \cong F(Z + 1) - F(Z)$, which as an equivalence of structure types means that a ΔF structure on a set S is an F structure on a set consisting of either single elements, or the empty set (that is, an F structure on any set larger than or equal to than S, since we are simply not counting some of the points toward the cardinality), with the exception that it cannot be simply an F structure on S (that is, those on sets whose elements are just one-point sets are removed). This means a ΔF-structure on S is an F structure on any set bigger than S.[1]
   What we're saying here $\Delta \cong E^A - 1$, is that the natural transformation Δ between structure types (which are functors) is the same as the nat. trans. which takes the derivative any number of times (other than 0) - that is, which adds any number of points surreptitiously into our set before putting the F-structure on it (where F is whatever structure Δ is acting on). This is obvious from the description in the last paragraph.

3. We want to say that if $\Delta G \cong F$, then $\sum_{i=0}^{n-1} F(i) \cong G(n) - G(0)$. The first says that F structures on S are G structures on anything strictly

[1] It seems there should be a correction in the cardinality accounting for the $k!$ denominator, something to the effect that all the added elements are interchangeable... Not clear to me at the moment what exactly this should be, though.

containing S. The second says that when we take the groupoid we get by evaluating G at the n-element set and removing from it the sub-groupoid which is the same as what we get evaluating G at the empty set, we should get the same as if we tack together all the groupoids obtained by evaluating F at sets of size smaller than n. When we proved this in the power series case, we had telescoping series - a similiar effect should occur here - each of the $F(i)$ groupoids will be easily describable as some groupoid $G(i+1)$ with $G(i)$ removed (the strict inclusion in our description of F), so taking all these together will fill in all the missing parts of $G(n)$ from $F(n-1)$ except the part where we evaluate at the empty set.[2]

4. Now we're defining an inverse to the derivative. This A^{-1} is clearly nonunique, since any given set S can be written in $|S|$ different ways as some smaller set with a single element adjoined. So when we take A^{-1} of some structure type F and put this new type on a set S, we get sets of F structures on S with one element removed (nonuniqueness coming from the fact that we could take out different elements, so there is no natural way to do this). This is not really a natural transformation of species, which presumably has something to do with the extra constant that comes in when we integrate (e.g. integrating the structure type "being a 5-element set", Z^5 gives $\frac{1}{6}Z^6$ - this fractional coefficient apparently counting the number of ways we could have done this, suggesting that it measures the degree of nonuniqueness of A^{-1}. That $AA^{-1}F \cong F$, if we swallow this problem and keep going, is due to the fact that putting an element in, once we have removed one, gives a set that can be naturally identified with the original by calling these the same element, which we are putting "back" in. That $A^{-1}AF$ is not naturally equivalent to F is due to the fact that if we remove the element AFTER putting one it, it may not be the same one.

5. Getting an inverse for Δ, the transformation which, applied to F gives F-structures on "bigger sets than S", is problematic for similar reasons to the problems we encountered in categorifying part 4, only more so. The "even more so" is visible in the fact that we would need to categorify the differential-operator power series we had for the difference operator Δ, which, however, has coefficients which are not only fractional (which we might could handle by some clever trick with groupoids), but also negative. This might could be handled by re-casting this whole crazy affair in the setting of cubical species, but I won't be doing that here.

Well, from here on in these problems will only get worse, so let's take this opportunity to stop categorifying for the moment, mentioning only that to do this properly would require working in a category in which we can handle negative coefficients, which by itself would be okay since such a category exists. We also also would have to somehow deal suitably with the non-naturality of

[2] I'm not quite sure how to put this better. Describing the groupoids involved here is still a bit mysterious to me. So it goes.

some of the transformations involved - and the method for doing this ought to give entities rather like structure types, but with fractional coefficients, where the denominators handle the size of the collection of different choices we might make at certain key points. Since we already have non-integral coefficients turning up when we consider groupoid cardinalities, and these are related to automorphisms of objects in a groupoid, this might be a relevant tool, using those non-unique choices to give those automorphisms. Then again, it might not.