Bernoulli numbers

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Let me start by pointing out that the spectrum of the algebra of entire functions is $\mathbf{C}$, while the spectrum of the algebra of formal power series is just $\{0\}$, because a formal power series is invertible if, and only if, its constant term is nonzero, so the only character of $\mathbf{C}[[z]]$ is the "evaluation at $z = 0$".

1. *Taylor's theorem.*

$$f(z+t) = \sum_{k\geq 0} \frac{t^k f^{(k)}(z)}{k!} = \sum_{k\geq 0} \frac{\big[(ta)^k f\big](z)}{k!} = \Big[\sum_{k\geq 0} \frac{(ta)^k}{k!}\Big] f(z) = (e^{ta} f)(z).$$

2. *First difference operator.*

$$(\Delta f)(z) = f(z+1) - f(z) = (e^a f)(z) - f(z) = (e^a - 1) f(z).$$

3. *Fundamental theorem of difference calculus.*

$$F(n) - F(0) = \sum_{i=0}^{n-1} \big[F(i+1) - F(i)\big] = \sum_{i=0}^{n-1} (\Delta F)(i) = \sum_{i=0}^{n-1} f(i).$$

4. *Integration is a one-sided inverse of differentiation.*

It is well-known that the derivative of an integral is the integrand, but the integral of a derivative may differ from the function to be differentiated by any constant. That is, $aa^{-1} = 1$ but $a^{-1}a \neq 1$.

5, 7, 8, 9. *Identities involving Bernoulli numbers.*

Clearly,

$$x = (e^x - 1)\frac{x}{e^x - 1} = \sum_{k\geq 1} \frac{x^k}{k!} \sum_{n\geq 0} B_n \frac{x^n}{n!}.$$

This means that

$$x = \sum_{m\geq 1} x^m \sum_{0\leq n<m} \frac{B_n}{n!(m-n)!},$$

so

$$B_0 = 1 \qquad \text{and} \quad 0 = \sum_{0\leq n<m} \frac{B_n}{n!(m-n)!} = \frac{1}{m!} \sum_{0\leq n<m} B_n \binom{m}{n},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 \\ 1 & 5 & 10 & 10 & 5 \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

so

$$B_0 = 1; \qquad B_1 = \frac{-1}{2}; \qquad B_2 = \frac{1}{6}; \qquad B_3 = 0; \qquad B_4 = \frac{-1}{30}.$$

Since the identity

$$x = \sum_{k\geq 1} \frac{x^k}{k!} \sum_{n\geq 0} B_n \frac{x^n}{n!}.$$

involves cancellations of finite sums, so

$$a = \sum_{k\geq 1} \frac{a^k}{k!} \sum_{n\geq 0} B_n \frac{a^n}{n!}.$$

for any operator a.

6. *The inverse of* Δ.

$$\Delta\Delta^{-1} = \Delta\frac{a}{e^a - 1}a^{-1} = aa^{-1} = 1.$$

Obviously, since $\Delta f = 0$ whenever f is constant, Δ cannot have a left-side inverse, so $\Delta^{-1}\Delta \neq 1$.

10.

Let $f(z) = z^4$. Then,

$$\sum_{i=0}^{n} i^4 = \sum_{i=0}^{n} f(i) = \sum_{i=0}^{n}(\Delta\Delta^{-1}f)(i) = (\Delta^{-1}f)(n+1) - (\Delta^{-1}f)(0).$$

But $(a^{-1}f)(z) = \frac{1}{5}z^5$, and

$$(\Delta^{-1}f)(z) = \frac{1}{5}\sum_{n\geq 0}\frac{B_n}{n!}\frac{d^n z^5}{dz^n} = \frac{1}{5}\sum_{n\geq 0} B_n\binom{5}{n}z^{5-n},$$

So

$$\sum_{i=0}^{n} i^4 = \frac{1}{5}\sum_{k\geq 1} B_k\binom{5}{k}(n+1)^{5-k} = \frac{1}{5}(n+1)^5 - \frac{1}{2}(n+1)^4 + \frac{1}{3}(n+1)^3 - \frac{1}{30}(n+1),$$

11. *A trick.*

$$\begin{aligned}1^p + \cdots + n^p =& \frac{1}{p+1}\sum_{k\geq 1}\binom{p+1}{k}B_k(n+1)^{p+1-k} = \frac{1}{p+1}\Big(\sum_{k\geq 0}\binom{p+1}{k}B_k(n+1)^{p+1-k} - (n+1)^{p+1}\Big) = \\ =& \frac{(B+n+1)^{p+1} - (n+1)^{p+1}}{p+1}\end{aligned}$$

if we interpret B_k to mean B^k. If we take $n = 0$ this reduces to

$$0 = (B+1)^{p+1} - B^{p+1}$$

which is the recursive formula for the Bernoulli numbers from part 8.