

## Bernoulli numbers

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Let me start by pointing out that the spectrum of the algebra of entire functions is  $\mathbf{C}$ , while the spectrum of the algebra of formal power series is just  $\{0\}$ , because a formal power series is invertible if, and only if, its constant term is nonzero, so the only character of  $\mathbf{C}[[z]]$  is the “evaluation at  $z = 0$ ”.

1. *Taylor’s theorem.*

$$f(z+t) = \sum_{k \geq 0} \frac{t^k f^{(k)}(z)}{k!} = \sum_{k \geq 0} \frac{[(ta)^k f](z)}{k!} = \left[ \sum_{k \geq 0} \frac{(ta)^k}{k!} \right] f(z) = (e^{ta} f)(z).$$

2. *First difference operator.*

$$(\Delta f)(z) = f(z+1) - f(z) = (e^a f)(z) - f(z) = (e^a - 1)f(z).$$

3. *Fundamental theorem of difference calculus.*

$$F(n) - F(0) = \sum_{i=0}^{n-1} [F(i+1) - F(i)] = \sum_{i=0}^{n-1} (\Delta F)(i) = \sum_{i=0}^{n-1} f(i).$$

4. *Integration is a one-sided inverse of differentiation.*

It is well-known that the derivative of an integral is the integrand, but the integral of a derivative may differ from the function to be differentiated by any constant. That is,  $aa^{-1} = 1$  but  $a^{-1}a \neq 1$ .

5, 7, 8, 9. *Identities involving Bernoulli numbers.*

Clearly,

$$x = (e^x - 1) \frac{x}{e^x - 1} = \sum_{k \geq 1} \frac{x^k}{k!} \sum_{n \geq 0} B_n \frac{x^n}{n!}.$$

This means that

$$x = \sum_{m \geq 1} x^m \sum_{0 \leq n < m} \frac{B_n}{n!(m-n)!},$$

so

$$B_0 = 1 \quad \text{and} \quad 0 = \sum_{0 \leq n < m} \frac{B_n}{n!(m-n)!} = \frac{1}{m!} \sum_{0 \leq n < m} B_n \binom{m}{n},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 \\ 1 & 5 & 10 & 10 & 5 \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

so

$$B_0 = 1; \quad B_1 = \frac{-1}{2}; \quad B_2 = \frac{1}{6}; \quad B_3 = 0; \quad B_4 = \frac{-1}{30}.$$

Since the identity

$$x = \sum_{k \geq 1} \frac{x^k}{k!} \sum_{n \geq 0} B_n \frac{x^n}{n!}.$$

involves cancellations of finite sums, so

$$a = \sum_{k \geq 1} \frac{a^k}{k!} \sum_{n \geq 0} B_n \frac{a^n}{n!}.$$

for any operator  $a$ .

6. The inverse of  $\Delta$ .

$$\Delta\Delta^{-1} = \Delta \frac{a}{e^a - 1} a^{-1} = aa^{-1} = 1.$$

Obviously, since  $\Delta f = 0$  whenever  $f$  is constant,  $\Delta$  cannot have a left-side inverse, so  $\Delta^{-1}\Delta \neq 1$ .

10.

Let  $f(z) = z^4$ . Then,

$$\sum_{i=0}^n i^4 = \sum_{i=0}^n f(i) = \sum_{i=0}^n (\Delta\Delta^{-1}f)(i) = (\Delta^{-1}f)(n+1) - (\Delta^{-1}f)(0).$$

But  $(a^{-1}f)(z) = \frac{1}{5}z^5$ , and

$$(\Delta^{-1}f)(z) = \frac{1}{5} \sum_{n \geq 0} \frac{B_n}{n!} \frac{d^n z^5}{dz^n} = \frac{1}{5} \sum_{n \geq 0} B_n \binom{5}{n} z^{5-n},$$

So

$$\sum_{i=0}^n i^4 = \frac{1}{5} \sum_{k \geq 1} B_k \binom{5}{k} (n+1)^{5-k} = \frac{1}{5}(n+1)^5 - \frac{1}{2}(n+1)^4 + \frac{1}{3}(n+1)^3 - \frac{1}{30}(n+1),$$

11. A trick.

$$\begin{aligned} 1^p + \dots + n^p &= \frac{1}{p+1} \sum_{k \geq 1} \binom{p+1}{k} B_k (n+1)^{p+1-k} = \frac{1}{p+1} \left( \sum_{k \geq 0} \binom{p+1}{k} B_k (n+1)^{p+1-k} - (n+1)^{p+1} \right) = \\ &= \frac{(B+n+1)^{p+1} - (n+1)^{p+1}}{p+1} \end{aligned}$$

if we interpret  $B_k$  to mean  $B^k$ . If we take  $n = 0$  this reduces to

$$0 = (B+1)^{p+1} - B^{p+1}$$

which is the recursive formula for the Bernoulli numbers from part 8.