Partition Functions, Partition Numbers and Making Change

John C. Baez, January 15, 2004

In class we studied the structure type G for which:

> *A G-structure on a finite set S is a way of chopping S into two parts, ordering the first part and chopping it into blocks of length 1, and ordering the second part and chopping it into blocks of length 2.*

Translating this description into an equation, we saw

$$G = \frac{1}{1-Z}\,\frac{1}{1-Z^2}$$

which gives the generating function

$$\begin{aligned} |G|(z) &= \tfrac{1}{1-z}\,\tfrac{1}{1-z^2} \\ &= (1+z+z^2+\cdots)(1+z^2+z^6+\cdots) \\ &= 1+z+2z^2+2z^3+3z^4+3z^5+\cdots \end{aligned}$$

If we write

$$|G|(z) = \sum_{n\geq 0} g_n z^n$$

then g_n is the number of ways of writing the number n as a sum of 1's and 2's, where we don't care about the order — or equivalently, where we write all the 1's first:

$$\begin{aligned} 0 &= \\ 1 &= 1 \\ 2 &= 1+1,\ 2 \\ 3 &= 1+1+1,\ 1+2, \\ 4 &= 1+1+1+1,\ 1+1+2,\ 2+2 \\ 5 &= 1+1+1+1+1,\ 1+1+1+2,\ 1+2+2 \end{aligned}$$

and so on. The reason we don't see an $n!$ in the denominator of this generating function is that putting a G-structure on the set n secretly involves choosing a total order on the elements of n, and there are $n!$ ways to make this choice.

(When the factors of $n!$ cancel like this, people often call the generating function an ordinary *generating function. When they don't, people call it an* exponential *generating functions. But, I'm trying to emphasize that ordinary generating functions are just exponential generating functions where the factors of $n!$ have cancelled.)*

Now let's generalize this idea in various directions!

1. Let a_n be the number of ways to write n as a sum of 1's, 5's, and 10's, where we don't care about the order. Write down a closed-form expression for

$$f(z) = \sum_{n\geq 0} a_n z^n$$

2. How many ways are there to make change for a hundred dollars in pennies, nickels and dimes? (Hint: in Mathematica, Series[f(z),z,n] computes the first n terms in the Taylor expansion of the function $f(z)$.)