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QUANTUM ~~GRAVITY~~ MECHANICS SEMINAR
ahem.

k-Colorings & Coherent States

1.) $C(k)_n := k^n$ as sets, so $|C(k)_n| = k^n$ as natural numbers.

2.) $$|C(k)|(z) := \sum_{n\geq 0} \frac{|C(k)_n|}{n!} z^n = \sum_{n \geq 0} \frac{(kz)^n}{n!} = e^{kz}.$$

3.) $$a|C(k)|(z) = \frac{d}{dz} e^{kz} = ke^{kz} = k|C(k)|(z).$$

4.) Suppose $s(z) = \sum_{n\geq 0} \frac{s_n z^n}{n!}$ is an eigenvector of a with eigenvalue σ.

Then
$$\frac{d}{dz} \sum_{n\geq 0} \frac{s_n z^n}{n!} = \sigma \sum_{n\geq 0} \frac{s_n z^n}{n!}$$
$$\sum_{n\geq 1} \frac{s_n z^{n-1}}{(n-1)!} = \sum_{n\geq 0} \frac{\sigma s_n z^n}{n!}$$
$$\sum_{n\geq 0} \frac{s_{n+1} z^n}{n!} = \sum_{n\geq 0} \frac{\sigma s_n z^n}{n!}$$

Equating corresponding coefficients we see that s_n must be recursively defined by $s_{n+1} = \sigma s_n$. Conversely if s_n has this property then
$$\frac{d}{dz} \sum_{n\geq 0} \frac{s_n z^n}{n!} = \sum_{n\geq 0} \frac{s_{n+1} z^n}{n!} = \sigma \sum \frac{s_n z^n}{n!}.$$

So the eigenvectors of a in $\mathbb{C}[[z]]$ are precisely all of the power series $\sum_{n\geq 0} \frac{s_n z^n}{n!}$ with $s_{n+1} = \sigma s_n$ for some $\sigma \in \mathbb{C}$. This means $s_n = s_0 \sigma^n$, so the eigenvectors of a in $\mathbb{C}[[z]]$ are just $s_0 e^{\sigma z}$, $s_0, \sigma \in \mathbb{C}$. The eigenvectors corresponding to k-colorings are just the special case of this where $\sigma = k$ and the overall factor s_0 is 1.

5.) To put an $AC(k)$-structure on n, we put a $C(k)$-structure on $n+1$. On the other hand, to put a $kC(k)$-structure on n, we split the set into two parts, where we put a k-structure on the first part and a $C(k)$-str. on the second.

But what is a k-structure? I don't think we've talked about it in the seminar yet, but the only thing that makes sense is that it's a structure that can be put only on the empty set, and can be put on the empty set in k distinct ways. ✓

So, to put a k-structure on n, we "split off" an empty subset $0 \subset n$, label that empty set by one of k colors, then we k-color $n - 0 = n$. But we might as well consider our empty set as just ~~another~~ element to k-color, so that a $kC(k)$ structure on n really is the same as (i.e. isomorphic to) a $C(k)$-structure on $n+1$. That is:

$$AC(k) \cong kC(k).$$ ✓

6.) Note first that $a^n a^{\dagger m} |1\rangle = \left| \frac{d^n z^m}{dz^n} \right\rangle = \begin{cases} |0\rangle & n > m \\ \frac{n!}{(m-n)!} |z^{m-n}\rangle & n \le m \end{cases}$

So:

$$\langle z^n | z^m \rangle = \langle 1 | a^n a^{\dagger m} | 1 \rangle = \begin{cases} \langle 1 | 0 \rangle = 0 & n > m \\ \frac{m!}{(m-n)!} \langle 1 | z^{m-n} \rangle & n \le m \end{cases}$$

But if $n < m$, then

$$\langle z^n | z^m \rangle = \langle z^m | z^n \rangle^* = \langle 1 | a^m a^{\dagger n} | 1 \rangle^* = \langle 1 | 0 \rangle^* = 0^* = 0.$$

So in fact,

$$\langle z^n | z^m \rangle = \begin{cases} 0 & n \neq m \\ n! & n = m \end{cases} = n!\,\delta_{n,m}.$$

7.) If $\psi(z) = \psi_0 e^{\sigma z} = \psi_0 \sum_{n\geq 0} \frac{\sigma^n z^n}{n!}$ is an eigenstate of a (i.e. a coherent state) then

$$\begin{aligned}
\langle \psi | \psi \rangle &= \psi_0^* \psi_0 \left\langle \sum_{n\geq 0} \frac{\sigma^n z^n}{n!} \,\middle|\, \sum_{m\geq 0} \frac{\sigma^m z^m}{m!} \right\rangle \\
&= \psi_0^* \psi_0 \sum_{n\geq 0} \left\langle \frac{\sigma^n z^n}{n!} \,\middle|\, \sum_{m\geq 0} \frac{\sigma^m z^m}{m!} \right\rangle \\
&= \psi_0^* \psi_0 \sum_{n\geq 0} \sum_{m\geq 0} \frac{\sigma^{*n} \sigma^m}{n!\,m!} \langle z^n | z^m \rangle \\
&= \psi_0^* \psi_0 \sum_{n\geq 0} \sum_{m\geq 0} \frac{\sigma^{*n} \sigma^m}{n!\,m!}\, n!\,\delta_{n,m} \\
&= \psi_0^* \psi_0 \sum_{n\geq 0} \frac{(\sigma^* \sigma)^n}{n!} \\
&= |\psi_0|^2 e^{|\sigma|^2}
\end{aligned}$$

Hence $\|\psi\| = |\psi_0| e^{|\sigma|^2/2}$,

and we get for a normalized coherent state:

$$\psi(z) = \frac{\psi_0}{|\psi_0|} e^{-|\sigma|^2/2} e^{\sigma z} = e^{it} e^{\sigma z - |\sigma|^2/2},$$

where $t \in \mathbb{R}$, or:

$$\boxed{\psi(z) = e^{\sigma z - |\sigma|^2/2 + it}}$$

& I'll settle for the case $t=0$ since a phase doesn't affect much!

8.) Expected values:

$$\langle\psi|q|\psi\rangle = \langle\psi|\frac{a+a^\dagger}{\sqrt{2}}|\psi\rangle$$
$$= \frac{1}{\sqrt{2}}\left[\langle\psi|a|\psi\rangle + \langle\psi|a^\dagger|\psi\rangle\right]$$
$$= \frac{1}{\sqrt{2}}\left[\langle\psi|a|\psi\rangle + \langle\psi|a|\psi\rangle^*\right]$$
$$= \sqrt{2}\,\mathrm{Re}\,\langle\psi|a|\psi\rangle$$
$$= \sqrt{2}\,\mathrm{Re}\,\langle\psi|\sigma|\psi\rangle$$
$$= \sqrt{2}\,\mathrm{Re}(\sigma\,\langle\psi|\psi\rangle)$$
$$= \sqrt{2}\,\mathrm{Re}\,\sigma$$

$$\langle\psi|p|\psi\rangle = \langle\psi|\frac{a-a^\dagger}{\sqrt{2}i}|\psi\rangle$$
$$= \frac{1}{\sqrt{2}i}\left[\langle\psi|a|\psi\rangle - \langle\psi|a^\dagger|\psi\rangle\right]$$
$$= \frac{1}{\sqrt{2}i}\,2i\,\mathrm{Im}\,\langle\psi|a|\psi\rangle$$
$$= \sqrt{2}\,\mathrm{Im}(\sigma\,\langle\psi|\psi\rangle)$$
$$= \sqrt{2}\,\mathrm{Im}\,\sigma.$$

✓

Since σ is an arbitrary complex number, it is clear we may design a state with $\langle p\rangle, \langle q\rangle$ equal to any real number we wish.

9.) $$\langle\psi|q^2|\psi\rangle = \langle\psi|\frac{a^2+aa^\dagger+a^\dagger a+a^{\dagger 2}}{2}|\psi\rangle$$
$$= \frac{1}{2}\left[\langle\psi|(a^2+[a,a^\dagger]+2a^\dagger a+a^{\dagger 2})|\psi\rangle\right]$$
$$= \frac{1}{2}\left[(\sigma^2+1+2\sigma^*\sigma+\sigma^{*2})\langle\psi|\psi\rangle\right]$$
$$= \frac{1}{2}\left[(\sigma+\sigma^*)^2+1\right]$$
$$= 2(\mathrm{Re}\,\sigma)^2+\frac{1}{2}$$

✓

Similarly,

$$\langle\psi|p^2|\psi\rangle = \langle\psi|\left(\frac{a-a^\dagger}{\sqrt{2}i}\right)^2|\psi\rangle$$

$$= -\frac{1}{2}\langle\psi|(a^2 - aa^\dagger - a^\dagger a + a^{\dagger 2})|\psi\rangle$$

$$= -\frac{1}{2}\langle\psi|(a^2 - [a,a^\dagger] - 2a^\dagger a + a^{\dagger 2})|\psi\rangle$$

$$= -\frac{1}{2}\left[\sigma^2 - 1 - 2\sigma^*\sigma + \sigma^{*2}\right]\langle\psi|\psi\rangle$$

$$= \frac{-1}{2}\left[(\sigma-\sigma^*)^2 - 1\right]$$

$$= \frac{-1}{2}(2i\,\mathrm{Im}\,\sigma)^2 + \frac{1}{2}$$

$$= 2(\mathrm{Im}\,\sigma)^2 + \frac{1}{2}.$$

✓

10.) Now

$$(\Delta_\psi p)^2 := \langle(p - \langle p\rangle)^2\rangle$$

$$= \langle p^2\rangle - \langle p\rangle^2$$

$$= \left(2(\mathrm{Im}\,\sigma)^2 + \tfrac{1}{2}\right) - (\sqrt{2}\,\mathrm{Im}\,\sigma)^2 = \frac{1}{2}$$

and

$$(\Delta_\psi q)^2 = \left(2(\mathrm{Re}\,\sigma)^2 + \tfrac{1}{2}\right) - (\sqrt{2}\,\mathrm{Re}\,\sigma)^2 = \frac{1}{2}$$

11.) $\Delta_\psi p = \Delta_\psi q = \frac{1}{\sqrt{2}}$, so $\Delta_\psi q\,\Delta_\psi p = \frac{1}{2}$.

✓

Note: we can only (currently) categorify coherent states where $\sigma \in \mathbb{N}$.