k-colourings

Miguel Carrión Álvarez

1. *k-coloring a finite set.*

A k-colouring of the set n is a function $f\colon n \to k$. For each element of n there are k choices of the image of f, and the choices for each of the elements are all independent, so there are k^n possible $f\colon n \to k$. Hence,

$$|C(k)_n| = k^n.$$

2. *The generating function of k-colourings.*

$$|C(k)|(z) = \sum_{n\geq 0} \frac{k^n}{n!} z^n = e^{kz}.$$

3, 4. *Annihilation.*

Trivially,

$$a|C(k)|(z) = \frac{d}{dz} e^{kz} = ke^{kz} = k|C(k)|(z).$$

Suppose that $|C(w)|(z)$ is an eigenvector of a with eigenvalue w, that is,

$$\frac{d}{dz}|C(w)|(z) = w|C(w)|(z).$$

Letting

$$|C(w)|(z) = \sum_{n\geq 0} \frac{|C(w)_n|}{n!} z^n,$$

the eigenvalue equation becomes

$$\sum_{n\geq 1} \frac{|C(w)_n|}{(n-1)!} z^{n-1} = \sum_{n\geq 0} \frac{w|C(w)_n|}{n!} z^n$$

so

$$\sum_{n\geq 0} \frac{|C(w)_{n+1}|}{n!} z^n = \sum_{n\geq 0} \frac{w|C(w)_n|}{n!} z^n$$

and $|C(w)_{n+1}| = w|C(w)_n|$. It follows that $|C(w)_n| = w^n|C(w)_0|$, and that

$$|C(w)|(z) = |C(w)_0|e^{wz},$$

so for each complex number w there is a one-dimensional space of eigenvectors of the annihilation operator with eigenvalue w.

5. *Categorified eigenvalue problem.*

Let K be the structure type such that putting it on a set S is "picking a colour out of k and S is empty". It follows that $|K|(z) = k$.

We seek a structure type T_k such that

$$AT_k \simeq K \times T_k.$$

Observe that "putting an AT_k structure on a set S" is "putting a T_k structure on the set $S+1$", and "putting a $K \times T_k$ structure on the set S" is the same as "picking a colour out of k and putting a T_k structure on S. That is, putting a T_k structure on $S+1$ is the same as picking a colour out of k and putting a T_k structure on S.

It easily follows by induction on $|S|$ that a T_k structure on the set S is a T_k structure on the empty set, and a k-colouring of S. How many ways there are to put a T_k structure on the empty set is undetermined, but it is basically the structure type K' for some integer k'. It follows that

$$T_k \simeq K' \times C(k).$$

6. *Inner product on Fock space.*

Assume, without loss of generality, that $n \geq m$. Then,

$$\langle z^n, z^m \rangle = \langle (a^*)^n 1, z^m \rangle = \langle 1, a^n z^m \rangle = \left\langle 1, \frac{d^n z^m}{dz^n} \right\rangle = \delta_{n,m} m! = \delta_{n,m} n!.$$

7. *Normalizing coherent states.*

Let $\psi_w = e^{wz}$. Then,

$$\langle \psi_w, \psi_w \rangle = \left\langle \sum_{n\geq 0} \frac{w^n}{n!} z^n, \sum_{n\geq 0} \frac{w^n}{n!} z^n \right\rangle = \sum_{n,m\geq 0} \frac{\bar{w}^n w^m}{n!m!} \langle z^n, z^m \rangle = \sum_{n\geq 0} \frac{|w|^{2n}}{n!} = e^{|w|^2}.$$

Hence, the normalized coherent state is

$$\psi_w = e^{-\frac{|w|^2}{2} + wz}$$

While we're at it, we are going to need $\|z\psi_w\|^2$ later:

$$\langle z\psi_w, z\psi_w \rangle = e^{-|w|^2} \sum_{n\geq 0} \frac{|w|^{2n}(n+1)}{n!} = |w|^2 + 1.$$

8,9,10,11. *Heisenberg uncertainty for coherent states.*

We use the facts that $q = \frac{1}{\sqrt{2}}(a + a^*)$ and that $p = \frac{1}{i\sqrt{2}}(a - a^*)$. Then,

$$\begin{aligned} \langle \psi_w, q\psi_w \rangle &= \frac{1}{\sqrt{2}} \big(\langle \psi_w, a\psi_w \rangle + \langle a\psi_w, \psi_w \rangle \big) = \frac{w + \bar{w}}{\sqrt{2}} \\ \langle \psi_w, p\psi_w \rangle &= \frac{w - \bar{w}}{\sqrt{2}i}. \end{aligned}$$

Similarly,

$$\begin{aligned} \langle \psi_w, q^2\psi_w \rangle &= \langle q\psi_w, q\psi_w \rangle = \frac{1}{2} \big(\langle a\psi_w, a\psi_w \rangle + \langle a^2\psi_w, \psi_w \rangle + \langle \psi_w, a^2\psi_w \rangle + \langle a^*\psi_w, a^*\psi_w \rangle \big) \\ &= \frac{1}{2} \big(|w^2| + \bar{w}^2 + w^2 + |w^2| + 1 \big) = \langle \psi_w, q\psi_w \rangle^2 + \frac{1}{2} \\ \langle \psi_w, p^2\psi_w \rangle &= \frac{1}{2} \big(|w^2| - \bar{w}^2 - w^2 + |w^2| + 1 \big) = \langle \psi_w, p\psi_w \rangle^2 + \frac{1}{2} \end{aligned}$$

It follows that $\sigma^2_{\psi_w} p = \sigma^2_{\psi_w} q = \frac{1}{2}$, as required.