QUANTUM GRAVITY HOMEWORK 2

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1. We have

$$!n = n! - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \dots$$

where A_i is the set of all permutations that fix i (the i^{th} element). Thus, we can equivalently consider A_i to be the set of permutations on the $(n-1)$-element set $n \setminus \{i\}$. So $|A_i| = (n-1)!$.

But which of the original n elements gets to play the role of i? There are $\binom{n}{1}$ possibilities in total. Since we are summing over all i,

$$\sum_i |A_i| = \binom{n}{1}(n-1)!.$$

BSA, $A_i \cap A_j$ corresponds to those permutations fixing both i and j. Thus $|A_i \cap A_j| = (n-2)!$ and since there are $\binom{n}{2}$ ways to choose i and j, we have

$$\sum_{i<j} |A_i \cap A_j| = \binom{n}{2}(n-2)!.$$

Continuing in this vein,

$$!n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \dots + (-1)^n \binom{n}{n}(n-n)!$$

2. Since $\binom{}{k=\frac{n!}{k!(n-k)!}},$ the above formula for $!n$ simplifies readily as

$$\begin{aligned} !n &= n! - \tfrac{n!}{1!(n-1)!}(n-1)! + \tfrac{n!}{1!(n-2)!}(n-2)! - \dots + (-1)^n \tfrac{n!}{1!(n-n)!}(n-n)! \\ &= n!\left(1 - \tfrac{1}{1!} + \tfrac{1}{2!} + \dots + \tfrac{(-1)^n}{n!}\right). \end{aligned}$$

3. The probability that nobody receives the correct coat is given by (number of derangements)/(number of permutations, i.e., the previous formula gives

$$\frac{!n}{n!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Thus,

$$\lim_{n\to\infty}\left(\frac{!n}{n!}\right) = \lim_{n\to\infty}\left(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}$$

4. Now $\lim_{n\to\infty}\left(!n\frac{e}{n!}\right)=1$ is equivalent to

$$!n \sim \frac{n!}{e}.$$

In other words,

$$\forall\varepsilon, \exists N \text{ s.t. } n \geq N \implies \left|!n - \frac{n!}{e}\right| < \varepsilon.$$

In particular, we can choose $\varepsilon = \frac{1}{2}$. Then $N = 1$. Since $!n$ is always an integer, and $\left|!n - \frac{n!}{e}\right|$ for $n \geq 1$, this shows $!n$ is the closest integer to $\frac{n!}{e}$.

Actually, that doesn't quite work because we need some info about the monotonicity of $\left|!n - \frac{n!}{e}\right|$, so let's break out the big guns:
We have

$$\frac{!n}{n!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!} = \frac{1}{e} - \sum_{k>n}\frac{(-1)^n}{n!}.$$

So

$$!n = \frac{n!}{e} - n!\sum_{k>n}\frac{(-1)^n}{n!},$$

and we just need to show

$$\left|n!\sum_{k>n}\frac{(-1)^n}{n!}\right| < \frac{1}{2}.$$

This is a rapidly convergent alternating series, so the sum is trapped between any two consecutive partial sums:

$$n!\sum_{k=n+1}^{N}\frac{(-1)^k}{k!} \leq n!\sum_{k=n+1}^{\infty}\frac{(-1)^k}{k!} \leq n!\sum_{k=n+1}^{N+1}\frac{(-1)^k}{k!}$$

In particular, it's trapped between the second and third:

$$n!\left(\tfrac{(-1)^{n+1}}{(n+1)!} + \tfrac{(-1)^{n+2}}{(n+2)!}\right) \leq n!\sum_{k=n+1}^{\infty}\frac{(-1)^k}{k!} \leq n!\left(\tfrac{(-1)^{n+1}}{(n+1)!} + \tfrac{(-1)^{n+2}}{(n+2)!} + \tfrac{(-1)^{n+3}}{(n+3)!}\right)$$

$$\tfrac{(-1)^{n+1}}{n+1} + \tfrac{(-1)^{n+2}}{(n+1)(n+2)} \leq n!\sum_{k=n+1}^{\infty}\frac{(-1)^k}{k!} \leq \tfrac{(-1)^{n+1}}{n+1} + \tfrac{(-1)^{n+2}}{(n+1)(n+2)} + \tfrac{(-1)^{n+3}}{(n+1)(n+2)(n+3)}$$

Now we can take the absolute value of the left-hand side[1]:

$$\left|\tfrac{(-1)^{n+1}}{n+1} + \tfrac{(-1)^{n+2}}{(n+1)(n+2)}\right| = \left|\tfrac{1}{n+1} - \tfrac{1}{(n+1)(n+2)}\right| = \left|\tfrac{n+1}{(n+1)(n+2)}\right| = \left|\tfrac{1}{n+2}\right| \leq \tfrac{1}{3}, \forall n \geq 1.$$

[1] Since one of $\{(-1)^{n+1}, (-1)^{n+2}\}$ is 1 and the other is -1, and since we are taking the absolute value, we can arbitrarily let one be 1 and the other be -1. Hence the first equality.

Similarly for the right-hand side:

$$\begin{aligned}\left|\tfrac{(-1)^{n+1}}{n+1}+\tfrac{(-1)^{n+2}}{(n+1)(n+2)}+\tfrac{(-1)^{n+3}}{(n+1)(n+2)(n+3)}\right| &= \left|\tfrac{(n+2)(n+3)+1}{(n+1)(n+2)(n+3)}-\tfrac{n+3}{(n+1)(n+2)(n+3)}\right| \\ &= \left|\tfrac{n^2+4n+4}{(n+1)(n+2)(n+3)}\right| \\ &= \left|\tfrac{n+2}{(n+1)(n+3)}\right| \\ &\le \tfrac{3}{8}, \forall n \ge 1\end{aligned}$$

Now the sum in question is trapped between two quantities of absolute value less than $\frac{1}{2}$ (and note that any two consecutive partial sums are less than $\frac{1}{24}$ apart), we have

$$\left| n! \sum_{k>n} \frac{(-1)^n}{n!} \right| < \frac{1}{2} \ ,$$

and hence

$$\left| !n - \frac{n!}{e} \right| < \frac{1}{2} \qquad \Longrightarrow \qquad !n = \left[\frac{n!}{e} \right].$$

5. We construct an isomorphism $P \cong E^Z D$.

If f is a permutation on S, then let

$$A = \{x \in S \vdots f(x) = x\}, \quad B = S \backslash A.$$

Now $f(b) \neq b, \forall b \in B$, by definition of B, so f is a derangement of B and the identity on A. In more categorical terms, f induces a splitting of S into two parts such that one part is left untouched and the other part is deranged. I.e., the first part is simply given the structure of a finite set, while the second part is given a derangement. This process of splitting a set into two pieces and putting different structures on each piece corresponds to multiplication of structure types. Since P is the structure type of "being permuted" and E^Z is the structure type of "being a finite set" and D is the structure type of "being deranged", we have

$$P \cong E^Z D.$$

6. Decategorifying the above isomorphism, we obtain

$$\frac{1}{1-z} = e^z |D|.$$

The left side comes from

$$\begin{aligned}|P|(z) &= \tfrac{p_0}{0!} + \tfrac{p_1}{1!} z^1 + \tfrac{p_2}{2!} z^2 + \tfrac{p_3}{3!} z^3 + \dots \\ &= \tfrac{0!}{0!} + \tfrac{1!}{1!} z^1 + \tfrac{2!}{2!} z^2 + \tfrac{3!}{3!} z^3 + \dots \\ &= 1 + z + z^2 + z^3 + \dots \\ &= \tfrac{1}{1-z},\end{aligned}$$

where the second line follows because there are $n!$ permutations of the n-element set, and the last line follows as a geometric series. Multiplying both sides by e^{-z} gives a formula for $|D|$:

$$|D|(z) = \frac{e^{-z}}{1-z}.$$

7. If we differentiate the above formula, the quotient rule yields

$$\begin{aligned}\tfrac{d}{dz}|D|(z) = \tfrac{d}{dz}\left(\frac{e^{-z}}{1-z}\right) &= \frac{-(1-z)e^{-z} - e^{-z}(-1)}{(1-z)^2} \\ &= e^{-z}\frac{1-(1-z)}{(1-z)^2} \\ &= e^{-z}\frac{z}{(1-z)^2}\end{aligned}$$

Thus, $(1-z)\frac{d}{dz}|D|(z) = e^{-z}\frac{z}{1-z}$. On the other hand,

$$|D|(z) - e^{-z} = \frac{e^{-z}}{1-z} - \frac{(1-z)e^{-z}}{1-z} = e^{-z}\frac{1-(1-z)}{1-z} = e^{-z}\frac{z}{1-z},$$

showing that

$$(1-z)\tfrac{d}{dz}|D|(z) = |D|(z) - e^{-z}.$$

8. Since the number of derangements of the n-element set is $!n$, we have

$$|D|(z) = \sum_{n=0}^{\infty} \frac{!n}{n!} z^n.$$

Also, from 7 we have

$$(1-z)\tfrac{d}{dz}|D|(z) = |D|(z) - e^{-z}.$$

Now, throwing caution to the wind and differentiating infinite sums term-by-term,

$$\begin{aligned}(1-z)\tfrac{d}{dz}|D|(z) &= (1-z)\sum_{n=1}^{\infty}\frac{!n}{n!}nz^{n-1} && \text{by above} \\ &= \sum_{n=1}^{\infty}\frac{!n}{n!}nz^{n-1} - \sum_{n=1}^{\infty}\frac{!n}{n!}nz^n && \text{distribute} \\ &= \sum_{n=0}^{\infty}\frac{!(n+1)}{(n+1)!}(n+1)z^n - \sum_{n=1}^{\infty}\frac{!n}{n!}nz^n && \text{reindex} \\ &= \sum_{n=1}^{\infty}\left(\frac{!(n+1)(n+1)}{(n+1)!} - \frac{(!n)n}{n!}\right)z^n && !1 = 0\end{aligned}$$

Now we manipulate the other side of the equation:

$$
\begin{aligned}
|D|(z) - e^{-z} &= \sum_{n=0}^{\infty} \frac{!n}{n!} z^n - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n && \text{by above} \\
&= \sum_{n=0}^{\infty} \frac{!n - (-1)^n}{n!} z^n && \text{combine} \\
&= \sum_{n=1}^{\infty} \frac{!n - (-1)^n}{n!} z^n && !0 - (-1)^0 = 0
\end{aligned}
$$

Combining these equalities gives

$$\sum_{n=1}^{\infty} \left(\frac{!(n+1)(n+1)}{(n+1)!} - \frac{(!n)n}{n!} \right) z^n = \sum_{n=1}^{\infty} \frac{!n - (-1)^n}{n!} z^n,$$

so equating the coefficients gives

$$\frac{!(n+1)(n+1)}{(n+1)!} - \frac{(!n)n}{n!} = \frac{!n - (-1)^n}{n!}.$$

Multiplying by $n!$ and cancelling the $n+1$, we get

$$
\begin{aligned}
!(n+1) - (!n)n &= !n - (-1)^n \\
!(n+1) &= (!n)n + !n - (-1)^n \\
!(n+1) &= !n(n+1) + (-1)^{n+1}
\end{aligned}
$$

There is another way to obtain the same result using just combinatorics. It is much more basic, but avoids possible irksome analysis technicalities. Note that an n-derangement can be derived from its predecessors in just one of two ways:

case i) Take a derangement of the first $n-1$ elements, then swap the n^{th} with one of them.

case ii) Derange $n-2$ of the first $n-1$ elements, then swap the n^{th} with the one that has remained hitherto fixed.

A moment's reflection shows that these are all the n-derangements, and one produced one way cannot be produced the other way. Since there are $n-1$ ways to do each of these things,

$$
\begin{aligned}
!n &= (n-1)\cdot!(n-1) + (n-1)\cdot!(n-2) \\
&= (n-1)(!(n-1) + !(n-2)) \\
&= n\cdot!(n-1) - (!(n-1) - (n-1)\cdot!(n-2)) \\
!n - n\cdot!(n-1) &= -(!(n-1) - (n-1)\cdot!(n-2))
\end{aligned}
$$

Note that if the left side of this last equation were denoted L_n, then the right side would be $-L_{n-1}$. This leads to a bizarre but simple reductio ad iteratum:

$$\begin{aligned} !n - n\cdot!(n-1) &= -(-(!(n-2) + (n-2)\cdot!(n-3))) \\ &= (-1)^k(!(n-k) + (n-k)\cdot!(n-k-1))) && \text{(after } k \text{ steps)} \\ &= (-1)^{n-2}(!2 + 2\cdot!1)) && \text{(let } k = n-2) \\ &= (-1)^n(1+0) && (d_1 = 0, d_2 = 1) \end{aligned}$$

Finally, adding back the $n\cdot!(n-1)$ gives

$$!n = n\cdot!(n-1) + (-1)^n.$$

9. We calculate $!n$ using Mathematica:

In[1] := $d1[n\_] := n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
In[2] := Table[$d1[n], n, 1, 10$]
Out[2]= $\{0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961\}$

In[3] := $d2[n\_] :=$ Round $\left[\frac{n!}{e}\right]$
In[4] := Table[$d2[n], \{n, 1, 10\}$]
Out[4]= $\{0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961\}$

In[5] := $d3[n\_] := d3[n] = nd3[n-1] + (-1)^n$
In[6] := $d3[0] = 1$
In[7] := Table[$d3[n], \{n, 1, 10\}$]
Out[7]:= $\{0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961\}$