

Deranged Assignment

Jeff Morton

1. If we have that A_i is the set of permutations of $\{1, \dots, n\}$ fixing i , then clearly $A_i \cong S_{n-1}$, acting on $\{1, \dots, i-1, i+1, \dots, n\}$. Similarly, an intersection of some of the A_i , say

$A_I = \bigcap_{i \in I} A_i$, is a permutation group which acts freely on $\{1, \dots, n\} - I$, so if $|I| = k$ then $A_I \cong S_{n-k}$, so that $|A_I| = (n-k)!$ and since there are $\binom{n}{k}$ ways to pick such an I , the summed cardinalities over all I of size k is $\binom{n}{k}(n-k)!$. This includes permutations which are in more than one A_I , however, so in fact we have, by the inclusion-exclusion principle, that

$$|D_n| = |S^n - \bigcup_{i=1}^n A_i| = |S^n| - \sum_{|I|=1} |A_I| + \sum_{|I|=2} |A_I| - \dots + (-1)^n \sum_{|I|=n} |A_I|$$

And by the above, we have:

$$!n = |D_n| = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \dots + (-1)^n \binom{n}{n}(n-n)!$$

2. In the above, we can put $\binom{n}{k}(n-k)! = \frac{n!}{k!(n-k)!}(n-k)! = \frac{n!}{k!}$. So the above expression is just

$$!n = n! - \frac{n!}{1!} + \frac{n!}{2!} - \dots + (-1)^n \frac{n!}{n!} = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!} \right)$$

3. We are looking for the probability that a random permutation is a derangement (since the coats are permuted among wearers). This

$$\frac{!n}{n!} = \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!} \right) = \sum_{k=0}^n \frac{(-1)^k}{k!}$$

But then taking the limit, we have $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(-1)^k}{k!} = e^{-1} = \frac{1}{e}$, since this is just the power-series expansion of e^x (which always converges) evaluated at $x = -1$.

4. To see that $!n$ is actually the *closest* integer to $\frac{n!}{e}$, note that

$$\frac{n!}{e} = n! \left(\frac{1}{e} \right) = n! \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \right),$$

whereas $!n$ is the finite sum with upper limit n , so the difference between them is the (necessarily convergent) infinite sum of the remaining terms:

$$!n - \frac{n!}{e} = n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}.$$

But this is an alternating series, the magnitudes of whose terms are strictly decreasing but all nonzero, so the size of the sum is strictly less than the size of the first term, which is $\frac{1}{n+1}$. If $n \geq 1$, then clearly we have that $\frac{!n}{n!}$ is the closest integer to $\frac{1}{e}$ since the difference between them is less than $\frac{1}{2}$.¹

5. If a D -structure on a set S is a derangement of the elements of S and a P -structure is any permutation, then suppose σ is a P -structure (a permutation). Then there is some subset of S , say F , consisting of all points fixed by σ , and on the remainder of S , σ acts as a derangement (since all the fixed

¹In the case $n = 0$, this fails, however, since $\frac{0!}{e} = \frac{1}{e}$, and the closest integer is 0, but $!0 = 1$, as explained in the post-script to part 9.

points are in F). Thus, any permutation - that is, any P -structure) on S gives a way to chop S into two pieces ($F \subseteq S$, and the rest) and a derangement on the rest, namely $\sigma|_{S-F}$. So we have a map $\psi : \sigma \mapsto (F(\sigma), \sigma|_{S-F(\sigma)})$. But moreover, if we choose any subset $F \subseteq S$ and put a derangement σ' on $S - F$, this gives a unique permutation σ whose action on S is to leave the elements of F fixed and act on $S - F$ just as σ' does. So in fact there is a bijection between P -structures, and the structure consisting of: chopping S into two disjoint subsets, and putting the structure of being a finite set (a E^Z -structure) on one (namely F) and putting a D -structure (derangement) on the other. So $P \cong E^Z \cdot D$ by this isomorphism ψ .

6. We have just seen that $P \cong E^Z \cdot D$, and so by general properties of generating functions, we have that $|P| = |E^Z| \cdot |D|$. Now, $|E^Z| = e^z$, and $|P| = \frac{1}{1-z}$, since P is the structure type of permutations (hence $|P_n| = n!$). Thus, we have: $\frac{1}{1-z} = e^z \cdot |D|(z)$. Since this is true as a statement about generating functions (no longer just about structure types), we can do all the usual algebraic operations, so in fact $|D|(z) = \frac{e^{-z}}{1-z}$.

7. From the fact that $|D|(z) = \frac{e^{-z}}{1-z}$, we find that

$$\begin{aligned} (1-z) \frac{d}{dz} |D|(z) &= (1-z) \left(\frac{(-1)(1-z)e^{-z} - (-1)e^{-z}}{(1-z)^2} \right) \\ &= -e^{-z} + \frac{e^{-z}}{1-z} \\ &= |D|(z) - e^{-z} \end{aligned}$$

8. We know by definition that $|D|(z)$ is the generating function for derangements, hence

$$|D|(z) = \sum_{n \geq 0} \frac{!n}{n!} z^n$$

and also that the relation found in part 7 is true. This means that

$$(1-z) \frac{d}{dz} |D|(z) + e^{-z} = |D|(z)$$

which we can use to get a recurrence relation on coefficients:

$$\begin{aligned}
\sum_{n \geq 0} \frac{!n}{n!} z^n &= (1-z) \frac{d}{dz} \sum_{n \geq 0} \frac{!n}{n!} z^n + \sum_{n \geq 0} \frac{(-z)^n}{n!} \\
&= \sum_{n \geq 1} n \cdot \frac{!n}{n!} (z^{n-1} - z^n) + \frac{(-1)^n z^n}{n!} && \text{(Reindex since derivative of } z^0 \text{ is 0)} \\
&= \sum_{n \geq 1} \left(\frac{!n}{(n-1)!} (z^{n-1} - z^n) + \frac{(-1)^n z^n}{n!} \right) \\
&= \sum_{n \geq 1} \left(\frac{!(n+1) + (-1)^n}{n!} - \frac{!(n)}{(n-1)!} \right) z^n && \text{(Gathering terms with common powers)} \\
&= \sum_{n \geq 1} \frac{!(n+1) + (-1)^n - n!(n)}{n!} z^n
\end{aligned}$$

Equating coefficients here gives that, (for $n \geq 1$),

$$\begin{aligned}
!(n+1) &= !n + n(!n) - (-1)^n \\
&= (n+1)(!n) + (-1)^{n+1}
\end{aligned}$$

which is what we wanted.

9. The first 6 values look like this:

n	$!n = \left\lfloor \frac{n!}{e} \right\rfloor$	$!n = n \cdot (n-1) + (-1)^n$
1	$\left\lfloor \frac{1!}{e} \right\rfloor = \lfloor 0.36788 \rfloor = 0$	0 (seed value)
2	$\left\lfloor \frac{2!}{e} \right\rfloor = \lfloor 0.73576 \rfloor = 1$	$2 \cdot 0 + 1 = 1$
3	$\left\lfloor \frac{3!}{e} \right\rfloor = \lfloor 2.20728 \rfloor = 2$	$3 \cdot 1 - 1 = 2$
4	$\left\lfloor \frac{4!}{e} \right\rfloor = \lfloor 8.82911 \rfloor = 9$	$4 \cdot 2 + 1 = 9$
5	$\left\lfloor \frac{5!}{e} \right\rfloor = \lfloor 44.14553 \rfloor = 44$	$5 \cdot 9 - 1 = 44$
6	$\left\lfloor \frac{6!}{e} \right\rfloor = \lfloor 264.87320 \rfloor = 265$	$6 \cdot 44 + 1 = 265$

Interestingly, the recurrence also holds true if we work backwards to $n = 0$: for then we get that $!1 = 0 = !0 - 1$, by the recurrence relation. So $!0 = 1$, which makes sense, since the only permutation of the empty set is a derangement simply because there are no points at all, and therefore no fixed points.