

Counting Partitions

John C. Baez, January 15, 2004

In class we studied the structure type G for which:

A G -structure on a finite set S is a way of chopping S into two parts, totally ordering the first part and chopping it into blocks of length 1, and totally ordering the second part and chopping it into blocks of length 2.

Translating this description into an equation, we saw

$$G = \frac{1}{1-Z} \frac{1}{1-Z^2}$$

which gives the generating function

$$\begin{aligned} |G|(z) &= \frac{1}{1-z} \frac{1}{1-z^2} = (1+z+z^2+\dots)(1+z^2+z^4+\dots) \\ &= 1+z+2z^2+2z^3+3z^4+3z^5+\dots \end{aligned}$$

If we write

$$|G|(z) = \sum_{n \geq 0} g_n z^n$$

then g_n is the number of ways of writing the number n as a sum of 1's and 2's, where we don't care about the order — or equivalently, where we write all the 1's first:

$$\begin{array}{lll} 0 & = & \implies g_0 = 1 \\ 1 & = & 1 \implies g_1 = 1 \\ 2 & = & 1+1, 2 \implies g_2 = 2 \\ 3 & = & 1+1+1, 1+2 \implies g_3 = 2 \\ 4 & = & 1+1+1+1, 1+1+2, 2+2 \implies g_4 = 3 \\ 5 & = & 1+1+1+1+1, 1+1+1+2, 1+2+2 \implies g_5 = 3 \end{array}$$

and so on. The reason we don't see an $n!$ in the denominator of this generating function is that putting a G -structure on the set n secretly involves choosing a total order on the elements of n , and there are $n!$ ways to make this choice.

(By the way: when the factors of $n!$ cancel for this reason, people call the generating function an **ordinary** generating function. When they don't cancel, people call it an **exponential** generating function. People often treat these cases as if they were very different. But they're not: ordinary generating functions are just exponential generating functions where a cancellation occurred. So, I'm not using this terminology. But, you should know it.)

Now, let's generalize this idea in various directions....

1. Let p_n be the number of ways to write n as a sum of 1's, 5's, and 10's, where we don't care about the order. Write down a closed-form expression for

$$p(z) = \sum_{n \geq 0} p_n z^n$$

2. How many different ways are there to make change for ten dollars in pennies, nickels and dimes? *Hint: in Mathematica, `Series[p[z], {z, c, n}]` computes the first n terms in the Taylor expansion of the function $p(z)$ about the point $z = c$.*

3. More generally, let p_n be the number of ways to write n as a sum of positive natural numbers chosen from some set $S \subseteq \mathbb{N}^+$, where we don't care about the order and we can use each number in S as many times as we like. Write a formula for

$$p(z) = \sum_{n \geq 0} p_n z^n$$

as a product of terms, one for each element of S .

4. Categorify your answer to the previous problem! In other words: describe a structure type P such that $|P| = p$, where p is the power series in problem 3. Show that the formula you wrote for p comes from an isomorphism between P and a product of structure types, one for each element of S .

5. Let q_n be the number of ways to write n as a sum of positive natural numbers chosen from the set $S \subseteq \mathbb{N}^+$, where we don't care about the order and *we can use each number in S at most once*. Write down an expression for

$$q(z) = \sum_{n \geq 0} q_n z^n$$

as a product of terms, one for each element of S .

6. Categorify your answer to the previous problem.

7. Using problems 3 and 5 and a bit of algebra, show that the number of ways of writing n as a sum of odd numbers is equal to the number of ways of writing it as a sum of distinct numbers! (Here 'numbers' means 'positive natural numbers'.)

Hint:

$$1 + z^n = \frac{1 - z^{2n}}{1 - z^n}.$$

The result in problem 7 is one of the simplest and most beautiful facts Leonard Euler discovered when he started playing around with generating functions in 1740. He came up with it when Phillipe Naude asked him how many ways there are to write a natural number as a sum of distinct parts. By this time he was already famous, but was beginning to go blind: his reply to Naude apologized for the delay caused by "bad eyesight for which I have been suffering for some weeks."

*For extra credit, give a **bijective proof** of this result: that is, construct an isomorphism between the set of ways of writing n as a sum of odd numbers and the set of ways of writing it as a sum of distinct numbers. You could try to categorify the calculation in problem 7, or try something else....*

There are many proofs of this beautiful fact. A more recent one, due to Victor Kac, uses quantum field theory. The generating function for 'writing a number as a sum of odd parts' is the partition function of a bosonic quantum field theory in 2d spacetime, while the generating function for 'writing a number as a sum of distinct parts' is the partition function for a fermionic quantum field theory in 2d spacetime. Using a trick called 'bosonization', which only works in 2d spacetime, we can show these quantum field theories are isomorphic — so their partition functions are the same!

We'll talk more about this kind of thing later. For details, try:

- Victor Kac, *Vertex Algebras for Beginners*, 2nd edition, American Mathematical Society, Providence, Rhode Island, 1998.