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Quantum Gravity Seminar
Homework #3 – Partitions

1.) p_n may be viewed as the number of ways of breaking up an ordered set into blocks of length 1,5, or 10, where all of the blocks of length 1 come first, followed by all blocks of length 5, and then the ones of length 10. This suggests we categorify p_n as a structure type P in the following way:

> A P-structure on a finite set S is a way of chopping S into three parts, putting the structure "totally ordered set" on the first part, putting the structure "totally ordered set of cardinality $\equiv 0 \pmod 5$" on the second part, and the structure "totally ordered set of cardinality $\equiv 0 (\text{mod } 10)$" on the third part.

That is, P is the following product of structure types:

$$P \cong \frac{1}{1-Z}\frac{1}{1-Z^5}\frac{1}{1-Z^{10}}.$$

Decategorifying, we get a closed form expression for the generating function:

$$p(z) = \frac{1}{1-z}\frac{1}{1-z^5}\frac{1}{1-z^{10}}.$$

2.) The sequence p_n goes like this:

$1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 6, 6, 6, 6, 6, 9, 9, 9, 9, 9, 12, 12, 12, 12, 12, 16, \ldots$

Note the pattern: it starts with $p_0 = 1$ and goes up every five places, first up by 1, then by 2, then 2, 3, 3, 4, 4, 5, 5, and so on. So some of the terms in the Taylor expansion of $p(z)$ are:

$$\begin{aligned}
p(z) = \frac{1}{1-z}\frac{1}{1-z^5}\frac{1}{1-z^{10}} =& 1 + z + z^2 + z^3 + z^4 \\
& + 2z^5 + 2z^6 + 2z^7 + 2z^8 + 2z^9 \\
& + 4z^{10} + 4z^{11} + 4z^{12} + 4z^{13} + 4z^{14} \\
& + 6z^{15} + 6z^{16} + 6z^{17} + 6z^{18} + 6z^{19} \\
& + 9z^{20} + 9z^{21} + 9z^{22} + 9z^{23} + 9z^{24} \\
& + 12z^{25} + 12z^{26} + 12z^{27} + 12z^{28} + 12z^{29} \\
& + 16z^{30} + \ldots \\
& \ldots + 10100z^{999} + 10201z^{1000} + \ldots
\end{aligned}$$

The coeffieient of z^{1000} tells us there are 10201 ways to give change for \$10 in cents, nickels and dimes.

3.) There's nothing special about 1, 5, and 10. The obvious generalization to the case of any set $S \subseteq \mathbb{N}^+$ is:

$$p(z) = \prod_{n \in S} \frac{1}{1 - z^n}.$$

4.) We want a structure type P that has p as its generating function. Since the categorification of a lower case letter is an upper case one, we get:

$$P = \prod_{n \in S} \frac{1}{1 - Z^n}.$$

Voilá! Of course, this actually makes complete sense: writing $n \in \mathbb{N}$ as a sum of numbers from S is the same as taking the set n, and chopping it up into ordered blocks with lengths in S, the smaller blocks always coming first to avoid counting permuted sums as different.

5.) & 6.) (not necessarily in that order) Given $n \in \mathbb{N}$ and $S \subseteq \mathbb{N}^+$ we could try writing n as a sum of distinct elements of $S \subseteq \mathbb{N}^+$, without regard to order, by using the following experimental approach. Take an ordered set of n elements and divide it into blocks with distinct lengths from S, adopting the arbitrary convention that the blocks increase in length as we go. After exhausting all of the possibilities, the number of ways we find that actually work is q_n. This suggests an obvious structure type Q, which has as its generating function

$$|Q|(z) = q(z) = \sum_{n \geq 0} q_n z^n.$$

More precisely, if we label the elements of S in increasing order, like this:

$$s_1 < s_2 < \cdots$$

then, to put a Q structure on a set X we first split off an ordered subset of X that is either empty or has cardinality s_1. Next, we split off an ordered subset of the remaining elements that is either empty or has cardinality s_2. Repeat this process for all of the elements of S.

Since $(1 + Z^s)$ is the structure type "empty set or ordered s-element set," Q is a product of such structure types, one for each element of S.

$$Q \cong \prod_{s \in S} (1 + Z^s).$$

Example: I can't resist mentioning one really cool example. In complex analysis we learn two ways of representing holomorphic functions – as Taylor series and as canonical products. One of the simplest cases to work out is the canonical product representation of the geometric series:

$$\prod_{n=0}^{\infty} (1 + z^{2^n}) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

On the left is the generating function of Q in the case where our set S of summands is $\{2^n | n \in \mathbb{N}\}$. On the right is the generating function for "total ordering of a set." The upshot is that in terms of generating functions, the above formula from complex analysis is just the statement that every natural number has exactly one binary expansion! That is, every natural number is uniquely the sum of zero or one 1's, zero or one 2's, zero or one 4's, ... , zero or one 2^k's, ...

Generalizing in slightly different direction, we get another formula for the same function – a formula I've never seen in complex analysis, but should be true nonetheless:

$$\frac{1}{1-z} = \prod_{n=0}^{\infty} (1 + z^{p^n} + z^{2p^n} + \cdots + z^{(p-1)p^n}) \quad \forall p \in \{2, 3, 4, \dots\}.$$

If your complex analysis teacher asks you to prove *this*, just write: "Proof: Any natural number has a unique p-ary expansion."

7.) The number of ways of writing n as a sum of odd positive integers is encoded as the nth (Maclauren) coefficient of the generating function

$$\prod_{n=1}^{\infty} \frac{1}{1-z^{2n-1}},$$

whereas the generating function for writing n as a sum of distinct positive integers is

$$\prod_{n=1}^{\infty}(1+z^n).$$

To see that these are equal, begin with the latter, use the 'hint,' and then separate the even and odd factors in the denominator. That is:

$$\prod_{n=1}^{\infty}(1+z^n) = \prod_{n=1}^{\infty} \frac{1-z^{2n}}{1-z^n} = \prod_{n=1}^{\infty} \frac{1-z^{2n}}{(1-z^{2n})(1-z^{2n-1})} = \prod_{n=1}^{\infty} \frac{1}{1-z^{2n-1}}.$$

This shows that the generating functions are equal, so the structure types at least have a chance of being isomorphic.

To see that writing n as a sum of odd numbers really is isomorphic to writing n as a sum of distinct numbers we would really like to find an explicit isomorphism. I got the following isomorphism not by any sophisticated method, but just by trying some examples to see if I could find a reversible algorithm. I seemed inevitably led to the following prescription.

Suppose I give you a representation of n as a sum of odd positive integers. Here's what to do with it:

1. Look at the sum. If the terms are all different, leave it alone! You're done! ("If it ain't broke, don't fix it.")
2. For each repeated term, replace two occurences by a single term – their sum. That is, replace

$$\underbrace{k + k + k + \cdots + k}_{m}$$

by

$$(2k) + \underbrace{k + k + k + \cdots + k}_{m-2}$$

3. `GOTO 1.`

Note that this process must terminate, since each time through the loop you are consolidating more terms – eventually, unless the loop ends sooner, you must end up with a single term: n itself.

The inverse process is easier to describe. If you give me a representation of n as a sum of distinct positive integers, here's what I'll do with it:

1. If all the integers in the sum are odd, I'll leave it be! I'm done!
2. Replace every term of the form $2k$ by two terms: $k + k$.
3. `GOTO 1.`

This procedure must also terminate – since I am splitting terms each time, I must eventually end up with $\underbrace{1 + 1 + \ldots + 1}_{n}$, unless I finish sooner.

I don't think I'm going to really bother quite *proving* that the two algorithms above are inverses. Instead, I'll just work out an example. The number 8 can be represented as a sum of odd integers or as a sum of distinct integers in exactly six ways. Following through the algorithms above we easily establish the following bijection.

odd integers		distinct integers
$1+1+1+1+1+1+1+1$	$\longleftrightarrow$	8
$3+1+1+1+1+1$	$\longleftrightarrow$	$4+3+1$
$5+1+1+1$	$\longleftrightarrow$	$5+2+1$
$7+1$	$\longleftrightarrow$	$7+1$
$5+3$	$\longleftrightarrow$	$5+3$
$3+3+2$	$\longleftrightarrow$	$6+2$