One-bonacci, Two-bonacci, Three-bonacci, Four...

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Just as the nth Fibonacci number counts the number of ways to take a totally ordered n-element set and chop it into blocks (substrings) of length 1 or 2, the nth **Tribonacci number***,* t_n*, counts the number of ways to take a totally ordered n-element set and chop it into blocks of length 1, 2, or 3. For example,* $t_4 = 7$ *since we have these 7 ways of chopping a 4-element totally ordered set into blocks of length 1, 2, or 3:*

$$\bullet|\bullet|\bullet|\bullet, \quad \bullet\bullet|\bullet|\bullet, \quad \bullet|\bullet\bullet|\bullet, \quad \bullet|\bullet|\bullet\bullet, \quad \bullet\bullet|\bullet\bullet, \quad \bullet\bullet\bullet|\bullet, \quad \bullet|\bullet\bullet\bullet.$$

We can also think of t_n *as counting the number of ways of writing n as a sum of 1's, 2's and 3's, where we keep track of the different orders of summation:*

$$4 = 1+1+1+1, \quad 2+1+1, \quad 1+2+1, \quad 1+1+2, \quad 2+2, \quad 3+1, \quad 1+3.$$

Let T be the structure type where a T-structure on the finite set S is a way of totally ordering S and then chopping it into blocks of length 1, 2, or 3. The number of ways to put such a structure on the n-element set is

$$|T_n| = n!\, t_n$$

since there are n! ways of ordering the set. The structure type T thus has the generating function

$$|T|(z) = \sum_{n=0}^{\infty} \frac{|T_n|}{n!} z^n = \sum_{n=0}^{\infty} t_n z^n$$

1. Copying the case of the Fibonacci numbers, find an isomorphism between the structure type T and some function involving T.

2. Decategorifying the above isomorphism, obtain an equation satisfied by the generating function $|T|$.

3. Use this equation to derive a recurrence relation for the Tribonacci numbers.

4. Solving the equation in part 2, find $|T|(z)$ as an explicit function of z.

5. Find a closed-form expression for the pole of $|T|$ closest to the origin. (If your algebra skills are rusty, feel free to use a computer algebra system such as Mathematica to solve the necessary cubic equation.)

6. Use part 5 and the souped-up version of Hadamard's theorem to show that

$$t_n \sim c\tau^n$$

for some constants $c, \tau > 0$, and find a closed-form expression for τ.

7. Use a computer algebra system to calculate the 100th Tribonacci number exactly, and also τ^{100}. Using these, estimate the constant c. *Extra Credit: guess or derive a closed-form expression for for c.*

8. Generalizing from the Fibonacci and Tribonacci cases, define the nth k-**bonacci number** to be the number of ways of taking a totally ordered n-element set and chopping it into blocks of

length $1, 2, \ldots, k$. What is the generating function for k-bonacci numbers? What recurrence do they satisfy?

9. Make a table of the nth k-bonacci number for $0 \leq n \leq 5$ and $1 \leq k \leq 5$. Ponder the pattern.

10. Define the nth ∞**-bonacci number** to be the number of ways of chopping a totally ordered n-element set into blocks of length $1, 2, 3, \ldots$ — that is, blocks of *arbitrary* positive integral length. What is the generating function for ∞-bonacci numbers? Find a simple closed-form expression.

11. Using the generating function for ∞-bonacci numbers, find a closed-form expression of the nth ∞-bonacci number.