

1. To put a Tribonacci structure (a T-structure) on a set  $S$ , we either put the structure "being the empty set" on  $T$  or we split the set into two parts, putting the structure "total ordering of a one, two, or three element set" on the first part, and then putting a T-structure on the second. In the notation of species:

$$T \cong 1 + (Z + Z^2 + Z^3)T$$

2. Decategorifying, the functors  $T, Z & 1$  become the functions  $|T|(z), z, & 1$ ; the isomorphism is lost, and " $\cong$ " becomes " $=$ ".

$$|T|(z) = 1 + (z + z^2 + z^3)|T|(z).$$

or,

$$|T|(z) = \frac{1}{1 - (z + z^2 + z^3)}$$

3. The Tribonacci numbers  $t_n$  are the coefficients in the power series expansion of  $|T|(z)$ :

$$|T|(z) = \sum_{n=0}^{\infty} t_n z^n.$$

So, using the above equation for  $|T|(z)$  we get

$$\begin{aligned} \sum_{n=0}^{\infty} t_n z^n &= 1 + (z + z^2 + z^3) \sum_{n=0}^{\infty} t_n z^n \\ \sum_{n=0}^{\infty} t_n z^n &= 1 + \sum_{n=0}^{\infty} t_n z^{n+1} + \sum_{n=0}^{\infty} t_n z^{n+2} + \sum_{n=0}^{\infty} t_n z^{n+3} \\ \sum_{n=0}^{\infty} t_n z^n &= 1 + \sum_{n=1}^{\infty} t_{n-1} z^n + \sum_{n=2}^{\infty} t_{n-2} z^n + \sum_{n=3}^{\infty} t_{n-3} z^n \end{aligned}$$

Equating corresponding coefficients, we find that for  $n \geq 3$ ,

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}$$

which is just the sort of recurrence relation we might expect for a generalization of the Fibonacci numbers.

4. Hmm... I already found the explicit description of  $|T|(z)$  in terms of  $z$ , above:

ok ✓

$$|T|(z) = \frac{1}{1-(z+z^2+z^3)}.$$

5.  $|T|(z)$  has poles at the roots of the cubic polynomial  $z^3+z^2+z-1$ . As we all know from high-school algebra (er... perhaps not!), these root can be found by first substituting  $z = \zeta - \frac{1}{3}$ . We get:

$$\left(\zeta - \frac{1}{3}\right)^3 + \left(\zeta - \frac{1}{3}\right)^2 + \left(\zeta - \frac{1}{3}\right) - 1 = 0$$

$$\left(\zeta^3 - \zeta^2 + \frac{1}{3}\zeta - \frac{1}{27}\right) + \left(\zeta^2 - \frac{2}{3}\zeta + \frac{1}{9}\right) + \left(\zeta - \frac{1}{3}\right) - 1 = 0$$

$$\zeta^3 + \frac{2}{3}\zeta - \frac{1}{27} + \frac{3}{27} - \frac{9}{27} - \frac{27}{27} = 0$$

$$\zeta^3 + \frac{2}{3}\zeta - \frac{34}{27} = 0,$$

leaving a cubic without second-order term. We next make the substitution  $\zeta = u+v$ , where the functional dependence of  $v$  on  $u$  is yet to be determined. This gives us:

$$(u+v)^3 + \frac{2}{3}(u+v) - \frac{34}{27} = 0$$

$$u^3 + 3u^2v + 3uv^2 + v^3 + \frac{2}{3}(u+v) - \frac{34}{27} = 0$$

$$u^3 + v^3 + 3(u+v)(uv + \frac{2}{9}) - \frac{34}{27} = 0$$

This is clearly simplified if we impose the constraint  $uv = -\frac{2}{9}$  on  $u$  and  $v$ , making the third term vanish. Using this to write  $v$  as a function of  $u$  we then get ✓

$$u^3 + \left(\frac{-2}{9u}\right)^3 - \frac{34}{27} = 0$$

or,

$$u^6 - \frac{34}{27}u^3 - \frac{8}{729} = 0$$

This last equation is quadratic in  $u^3$  so we can use the formula we really did learn in high school:

$$u^3 = \frac{\frac{34}{27} \pm \sqrt{\left(\frac{34}{27}\right)^2 + \frac{32}{729}}}{2}$$

$$\text{or } u^3 = \frac{17}{27} \pm \sqrt{\left(\frac{17}{27}\right)^2 + \frac{8}{729}}$$

If we choose the "+" sign in the above, then

$$\begin{aligned} v^3 &= \left(\frac{-2}{9u}\right)^3 = -\frac{\frac{-8}{729}}{\frac{17}{27} + \sqrt{\left(\frac{17}{27}\right)^2 + \frac{8}{729}}} \\ &= \frac{\frac{8}{729} \left( \frac{17}{27} - \sqrt{\left(\frac{17}{27}\right)^2 + \frac{8}{729}} \right)}{\left(\frac{17}{27}\right)^2 - \left[ \left(\frac{17}{27}\right)^2 + \frac{8}{729} \right]} \\ &= \frac{\frac{17}{27} - \sqrt{\left(\frac{17}{27}\right)^2 + \frac{8}{729}}}{\frac{8}{729}} = \frac{u^3}{u^3} \end{aligned}$$

So, we can write, by simplifying  $\sqrt{(\frac{17}{27})^2 + \frac{8}{729}}$  to  $\frac{\sqrt{33}}{9}$ :

$$u^3 = \frac{17}{27} + \frac{\sqrt{33}}{9} \quad v^3 = \frac{17}{27} - \frac{\sqrt{33}}{9}.$$

This gives us three values for the pair  $(u, v)$ , corresponding to the three branches of  $\sqrt[3]{\cdot}$  in  $\mathbb{C}$ :

$$u_1 = \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}}, \quad v_1 = \sqrt[3]{\frac{17}{27} - \frac{\sqrt{33}}{9}} \quad (\text{real roots!})$$

$$u_2 = e^{2\pi i/3} \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}}, \quad v_2 = e^{2\pi i/3} \sqrt[3]{\frac{17}{27} - \frac{\sqrt{33}}{9}}$$

$$u_3 = e^{4\pi i/3} \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}}, \quad v_3 = e^{4\pi i/3} \sqrt[3]{\frac{17}{27} - \frac{\sqrt{33}}{9}}$$

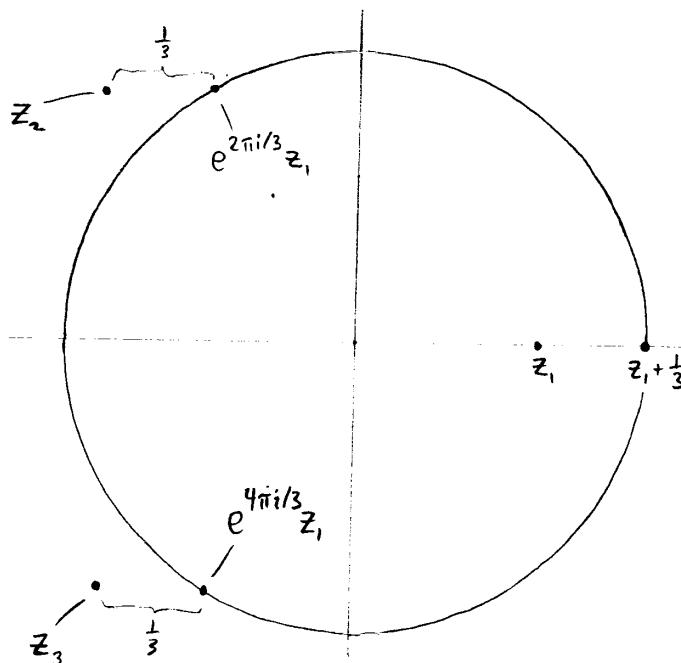
From these, we obtain our three roots via  $z = b - \frac{1}{3} = u + v - \frac{1}{3}$ .

$$z_1 = \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}} + \sqrt[3]{\frac{17}{27} - \frac{\sqrt{33}}{9}} - \frac{1}{3} = .543689012692\dots$$

$$z_2 = e^{2\pi i/3}(z_1 + \frac{1}{3}) - \frac{1}{3}$$

$$z_3 = e^{4\pi i/3}(z_1 + \frac{1}{3}) - \frac{1}{3}$$

We want the root closest to the origin. Geometrically, it is clear that this is  $z_1$ .



So we can finally answer question 5: The pole of  $|T(z)|$  closest to the origin is at

$$z = \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}} + \sqrt[3]{\frac{17}{27} - \frac{\sqrt{33}}{9}} - \frac{1}{3} = \frac{1}{\psi}$$

Where I guess  $\psi$  must be what's called the "silver ratio." It is that!

6. The "Souped-up Hadamard Theorem" says if  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  is analytic in  $N_R(0)$  except for a simple pole at radius  $r < R$  from the origin, then  $|a_n| \sim c \left(\frac{1}{r}\right)^n$  for some  $c > 0$ . This is just the situation we have:  $|T(z)|$  has only the pole at  $z_1 = \frac{1}{\psi}$  within the disc  $N_{z_1 + \frac{1}{3}}(0)$  so we have, for some  $c$ ,  $t_n \sim c \psi^n$ .

i.e.,  $\frac{t_n}{c \psi^n} \rightarrow 1 \text{ as } n \rightarrow \infty$



7. The Tribonacci numbers look like this:

$$t_0 = 1$$

$$t_1 = 1$$

$$t_2 = 2$$

$$t_3 = 4$$

$$t_4 = 7$$

.

.

$$t_{12} = 927$$

.

.

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$$t_{99} = 98,079,530,178,586,034,536,500,564$$

$$t_{100} = 180,396,380,815,100,901,214,157,639$$

So, one of these is  
the "hundredth",  
depending on whether  
you start with the  
0th or the 1st.



Also,

$$\varphi^{100} = \left(\frac{1+\sqrt{5}}{2}\right)^{100} = 29,170,531,916,412,951,080,3819,574.2\dots$$

So

$$\frac{t_n}{\varphi^n} \rightarrow c$$

$$\Rightarrow c \approx \frac{t_{100}}{\varphi^{100}} = .618419922\dots$$



this is close to the  
Golden ratio, but that can't be...  
quite right.

After solving that cubic, I'm out of energy for trying to get "Extra Credit."

8. I'll write  $B_k$  for the  $k$ -bonacci species, where "B" stands for "Bonacci" and the index  $k$  runs over the set  $1, F_1, T_1, 4, 5, 6, \dots$ .

By the same lines of reasoning used in the Fibonacci and Tribonacci cases, we have:

$$B_k \cong 1 + (z + z^2 + \dots + z^k) B_k$$

and hence

$$|B_k|(z) = \frac{1}{1 - z - z^2 - \dots - z^k}.$$

Also by the same methods, we find

$$\boxed{b_{k,n} = b_{k,n-1} + b_{k,n-2} + \dots + b_{k,n-k} \quad \forall n > k-1}$$

where

$$|B_k|(z) = \sum_{n=0}^{\infty} b_{k,n} z^n.$$

9. Here's a table of  $k$ -bonacci numbers (including the  $k=0$  case!)

<u><math>k=0</math></u>	<u><math>k=1</math></u>	<u><math>k=2</math></u>	<u><math>k=3</math></u>	<u><math>k=4</math></u>	<u><math>k=5</math></u>	$\dots$
1	1	1	1	1	1	$\dots$
0	1	1	1	1	1	
0	1	2	2	2	2	$\dots$
0	1	3	4	4	4	
0	1	5	7	8	8	
0	1	8	13	15	16	$\dots$

the  $\infty$ -bonacci  
numbers,  
 $b_{\infty,n}$

"Ways of chopping up an  
n-element ordered set  
into blocks of length at  
most zero."  $\approx 1$

10. To put an  $\infty$ -bonacci structure on a set, we [split the set in two parts, totally order the first part and then put an  $\infty$ -bonacci structure on the second] or [put the empty set str. on the set]:

$$B_\infty = \left( \sum_{n=1}^{\infty} z^n \right) B_\infty + 1$$

which, decategorified, gives us the generating function:

$$|B_\infty|(z) = \left( \sum_{n=1}^{\infty} z^n \right) |B_\infty|(z) + 1$$

$$|B_\infty|(z) \left( 1 - \sum_{n=1}^{\infty} z^n \right) = 1$$

$$|B_\infty|(z) = \frac{1}{1 - \sum_{n=1}^{\infty} z^n}$$

$$\begin{aligned} |B_\infty|(z) &= \frac{1}{1 - \frac{z}{1-z}} \\ &= \frac{1-z}{(1-z)-z} \\ &= \boxed{\frac{1-z}{1-2z}} \end{aligned}$$

Now we can write

$$\begin{aligned} |B_\infty|(z) &= \sum_{n=0}^{\infty} b_{\infty,n} z^n = \frac{1-z}{1-2z} = (1-z) \sum_{n=0}^{\infty} (2z)^n \\ &= \sum_{n=0}^{\infty} 2^n z^n - \sum_{n=1}^{\infty} 2^{n-1} z^n \end{aligned}$$

so we get, for  $n \geq 1$ ,  $b_{\infty,n} = 2^n - 2^{n-1} = (2-1)2^{n-1} = \boxed{2^{n-1}}$ ,

while for  $n=0$ ,  $b_{\infty,0} = 1$ . Note: this is just what we expect, since  $b_{\infty,n}$  is the number of ways to put any number of bars between  $n$  dots, e.g.  $\bullet | \bullet | \bullet \cdot \bullet | \bullet \cdot \cdot \cdot \cdot \cdot$ , ( $n$  dots)

$\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$  ( $n-1$ ) spaces  $\Rightarrow 2^{n-1}$  independent choices.