Excellent!

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Quantum Gravity Seminar
Homework #1

Bonacci

1. To put a Tribonacci structure (a T-structure) on a set S, we either put the structure "being the empty set" on T or we split the set into two parts, putting the structure "total ordering of a one, two, or three element set" on the first part, and then putting a T-structure on the second. In the notation of species:

$$\boxed{T \cong 1 + (Z + Z^2 + Z^3)T}$$

2. Decategorifying, the functors T, Z & 1 become the functions $|T|(z)$, z, & 1; the isomorphism is lost, and "$\cong$" becomes "$=$":

$$|T|(z) = 1 + (z + z^2 + z^3)|T|(z).$$

or,

$$|T|(z) = \frac{1}{1 - (z + z^2 + z^3)}$$

3. The Tribonacci numbers t_n are the coefficients in the power series expansion of $|T|(z)$:

$$|T|(z) = \sum_{n=0}^{\infty} t_n z^n .$$

So, using the above equation for $|T|(z)$ we get

$$\sum_{n=0}^{\infty} t_n z^n = 1 + (z + z^2 + z^3) \sum_{n=0}^{\infty} t_n z^n$$

$$\sum_{n=0}^{\infty} t_n z^n = 1 + \sum_{n=0}^{\infty} t_n z^{n+1} + \sum_{n=0}^{\infty} t_n z^{n+2} + \sum_{n=0}^{\infty} t_n z^{n+3}$$

$$\sum_{n=0}^{\infty} t_n z^n = 1 + \sum_{n=1}^{\infty} t_{n-1} z^n + \sum_{n=2}^{\infty} t_{n-2} z^n + \sum_{n=3}^{\infty} t_{n-3} z^n$$

Equating corresponding coefficients, we find that for $n \geq 3$,

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}$$

which is just the sort of recurrence relation we might expect for a generalization of the Fibonacci numbers.

4. Hmmmm... I already found the explicit description of $|T|(z)$ in terms of z, above:

ok

$$|T|(z) = \frac{1}{1-(z+z^2+z^3)} .$$

5. $|T|(z)$ has poles at the roots of the cubic polynomial z^3+z^2+z-1. As we all know from high-school algebra (er... perhaps not!), these root can be found by first substuting $z = \zeta - \frac{1}{3}$. We get:

$$\left(\zeta-\tfrac{1}{3}\right)^3 + \left(\zeta-\tfrac{1}{3}\right)^2 + \left(\zeta-\tfrac{1}{3}\right) - 1 = 0$$

$$\left(\zeta^3-\zeta^2+\tfrac{1}{3}\zeta-\tfrac{1}{27}\right) + \left(\zeta^2-\tfrac{2}{3}\zeta+\tfrac{1}{9}\right) + \left(\zeta-\tfrac{1}{3}\right) - 1 = 0$$

$$\zeta^3 + \frac{2}{3}\zeta - \frac{1}{27} + \frac{3}{27} - \frac{9}{27} - \frac{27}{27} = 0$$

$$\zeta^3 + \frac{2}{3}\zeta - \frac{34}{27} = 0,$$

leaving a cubic without second-order term. We next make the substitution $\zeta = u + v$, where the functional dependence of v on u is yet to be determined. This gives us:

$$(u+v)^3 + \frac{2}{3}(u+v) - \frac{34}{27} = 0$$

$$u^3 + 3u^2v + 3uv^2 + v^3 + \frac{2}{3}(u+v) - \frac{34}{27} = 0$$

$$u^3 + v^3 + 3(u+v)(uv+\frac{2}{9}) - \frac{34}{27} = 0$$

This is clearly simplified if we impose the constraint $uv = -\frac{2}{9}$ on u and v, making the third term vanish. Using this to write v as a function of u we then get ✓

$$u^3 + \left(\frac{-2}{9u}\right)^3 - \frac{34}{27} = 0$$

or,

$$u^6 - \frac{34}{27}u^3 - \frac{8}{729} = 0$$

This last equation is quadratic in u^3 so we can use the formula we really <u>did</u> learn in high school:

$$u^3 = \frac{\frac{34}{27} \pm \sqrt{\left(\frac{34}{27}\right)^2 + \frac{32}{729}}}{2}$$

$$\text{or} \quad u^3 = \frac{17}{27} \pm \sqrt{\left(\frac{17}{27}\right)^2 + \frac{8}{729}}$$

If we choose the "+" sign in the above, then

$$v^3 = \left(\frac{-2}{9u}\right)^3 = \frac{\frac{-8}{729}}{\frac{17}{27} + \sqrt{\left(\frac{17}{27}\right)^2 + \frac{8}{729}}}$$

$$= \frac{\cancel{\frac{-8}{729}}\left(\frac{17}{27} - \sqrt{\left(\frac{17}{27}\right)^2 + \frac{8}{729}}\right)}{\cancel{\left(\frac{17}{27}\right)^2} - \left[\cancel{\left(\frac{17}{27}\right)^2} + \cancel{\frac{8}{729}}\right]}$$

$$= \frac{17}{27} - \sqrt{\left(\frac{17}{27}\right)^2 + \frac{8}{729}} = \overline{u}^3$$

So, we can write, by simplifying $\sqrt{\left(\frac{17}{27}\right)^2 + \frac{8}{729}}$ to $\frac{\sqrt{33}}{9}$:

$$u^3 = \frac{17}{27} + \frac{\sqrt{33}}{9} \qquad v^3 = \frac{17}{27} - \frac{\sqrt{33}}{9} .$$

This gives us three values for the pair (u,v), corresponding to the three branches of $\sqrt[3]{\ }$ in $\mathbb{C}$:

$$u_1 = \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}} , \quad v_1 = \sqrt[3]{\frac{17}{27} - \frac{\sqrt{33}}{9}} \qquad \text{(real roots!)}$$

$$u_2 = e^{2\pi i/3} \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}} , \quad v_2 = e^{2\pi i/3} \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}}$$

$$u_3 = e^{4\pi i/3} \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}} , \quad v_3 = e^{4\pi i/3} \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}}$$

From these, we obtain our three roots via $z = \zeta - \frac{1}{3} = u + v - \frac{1}{3}$.

$$z_1 = \sqrt[3]{\frac{17}{27} + \frac{\sqrt{33}}{9}} + \sqrt[3]{\frac{17}{27} - \frac{\sqrt{33}}{9}} - \frac{1}{3} = .543689012692\ldots$$

$$z_2 = e^{2\pi i/3}\left(z_1 + \frac{1}{3}\right) - \frac{1}{3}$$

$$z_3 = e^{4\pi i/3}\left(z_1 + \frac{1}{3}\right) - \frac{1}{3}$$

7. The Tribonacci numbers look like this:

$$t_0 = 1$$

$$t_1 = 1$$

$$t_2 = 2$$

$$t_3 = 4$$

$$t_4 = 7$$

$$\vdots$$

$$t_{12} = 927$$

$$\vdots$$

$$t_{99} = 98{,}079{,}530{,}178{,}586{,}034{,}536{,}500{,}564$$

$$t_{100} = 180{,}396{,}380{,}815{,}100{,}901{,}214{,}157{,}639$$

So, one of these is the "hundredth", depending on whether you start with the 0th or the 1st.

✓

Also,

$$\tau^{100} = (\psi)^{100} = 29{,}170{,}531{,}916{,}412{,}951{,}080{,}3819{,}574.2\ldots$$

So

$$\frac{t_n}{\tau^n} \longrightarrow c$$

$$\Longrightarrow c \approx \frac{t_{100}}{\tau^{100}} = .61841992 2\ldots$$

this is close to the Golden ratio, but that can't be quite right.

no....

After solving that cubic, I'm out of energy for trying to get "Extra Credit."

8. I'll write B_k for the k-bonacci species, where "B" stands for "Bonacci" and the index k runs over the set

$$1, \mathrm{Fi}, \mathrm{Tri}, 4, 5, 6, \ldots .$$

By the same lines of reasoning used in the Fibonacci and Tribonacci cases, we have:

$$B_k \cong 1 + (Z + Z^2 + \cdots + Z^k) B_k$$

and hence

$$|B_k|(z) = \frac{1}{1 - z - z^2 - \cdots - z^k} .$$

Also by the same methods, we find

$$\boxed{b_{k,n} = b_{k,n-1} + b_{k,n-2} + \cdots + b_{k,n-k} \qquad \forall n > k-1}$$

where

$$|B_k|(z) = \sum_{n=0}^{\infty} b_{k,n} z^n .$$

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9. Here's a table of k-bonacci numbers (including the k=0 case!)

k=0	k=1	k=2	k=3	k=4	k=5	
1	1	1	1	1	1	...
0	1	1	1	1	1	
0	1	2	2	2	2	...
0	1	3	4	4	4	
0	1	5	7	8	8	
0	1	8	13	15	16	...
				⋮		

the ∞-bonacci numbers, $b_{\infty,n}$

"Ways of chopping up an n-element ordered set into blocks of length at most zero." $\cong 1$

10. To put an ∞-bonacci structure on a set, we [split the set in two parts, totally order the first part and then put an ∞-bonacci structure on the second] <u>or</u> [put the empty set str. on the set]:

$$B_\infty = \left(\sum_{n=1}^{\infty} Z^n\right) B_\infty + 1$$

which, decategorified, gives us the generating function:

$$|B_\infty|(z) = \left(\sum_{n=1}^{\infty} z^n\right) |B_\infty|(z) + 1$$

$$|B_\infty|(z)\left(1 - \sum_{n=1}^{\infty} z^n\right) = 1$$

$$|B_\infty|(z) = \frac{1}{1 - \sum_{n=1}^{\infty} z^n}$$

$$|B_\infty|(z) = \frac{1}{1 - \frac{z}{1-z}}$$

$$= \frac{1-z}{(1-z) - z}$$

$$= \boxed{\frac{1-z}{1-2z}}$$

Now we can write

$$|B_\infty|(z) = \sum_{n=0}^{\infty} b_{\infty,n} z^n = \frac{1-z}{1-2z} = (1-z)\sum_{n=0}^{\infty} (2z)^n$$

$$= \sum_{n=0}^{\infty} 2^n z^n - \sum_{n=1}^{\infty} 2^{n-1} z^n$$

So we get, for $n \geq 1$, $b_{\infty,n} = 2^n - 2^{n-1} = (2-1)2^{n-1} = \boxed{2^{n-1}}$, while for $n = 0$, $b_{\infty,0} = 1$. Note: this is just what we expect, since $b_{\infty,n}$ is the number of ways to put any number of bars between n dots, e.g. $\bullet|\bullet|\bullet\ \bullet|\bullet\ \bullet\ \bullet\ \bullet\ \bullet$ (n dots)

$(n-1)$ spaces $\Rightarrow 2^{n-1}$ independent choices.