# QUANTUM GRAVITY HOMEWORK 1

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1. An isomorphism between the structure type $T$ and a function involving $T$:

$$T \cong Z^3 T + Z^2 T + ZT + 1.$$

To put a $T$-structure on $S$, either:

(i) Hew it in twain and put the structure of being a totally ordered 3-elt set on the first part and a $T$-structure on the second, or

(ii) hew it in twain and put the structure of being a totally ordered 2-elt set on the first part and a $T$-structure on the second, or

(iii) hew it in twain and put the structure of being a totally ordered 1-elt set on the first part and a $T$-structure on the second, or

(iv) it is the empty set, so give it this structure.

2. By decategorifying the above isomorphism, we obtain $|T| = z^3|T| + z^2|T| + z|T| + 1$.

3. Using this relation, we find a recurrence relation for the Tribonacci numbers:

$$\begin{aligned}
|T| = \sum_{n\geq 0} t_n z^n &= z^3|T| + z^2|T| + z|T| + 1 \\
&= \sum_{n\geq 0} t_n z^{n+3} + \sum_{n\geq 0} t_n z^{n+2} + \sum_{n\geq 0} t_n z^{n+1} + 1 \\
&= \sum_{n\geq 3} t_n z^n + \sum_{n\geq 2} t_n z^n + \sum_{n\geq 1} t_n z^n + 1
\end{aligned}$$

Thus, $t_n = t_{n-3} + t_{n-2} + t_{n-1} \forall n \geq 3$. We can readily compute the first three terms by examining the structure on 0,1, and 2-elt sets:

$$\begin{array}{lll}
t_0 = 1 & & \leftarrow \varnothing \\
t_1 = 1 & \bullet & \\
t_2 = 2 & \bullet\bullet & \bullet|\bullet
\end{array}$$

4. Solving the equation in 2, we obtain $|T|(z)$ as an explicit function of $z$.

$$\begin{aligned}|T|(z) &= z^3|T|(z) + z^2|T|(z) + z|T|(z) + 1\\ |T|(z) - z|T|(z) - z^2|T|(z) - z^3|T|(z) &= 1\\ (1 - z - z^2 - z^3)|T|(z) &= 1\\ |T|(z) &= \frac{1}{1 - z - z^2 - z^3}\end{aligned}$$

5. Now we find a closed form expression for the poles of $|T|$. We need to find the roots of

$$-1 + z + z^2 + z^3 = 0,$$

so the coefficients are $a_0 = -1, a_1 = 1, a_2 = 1$ and we compute

$$p = a_1 - \frac{a_2^2}{3} = 1 - \frac{1}{3} = \frac{2}{3}, \text{ and}$$

$$q = \frac{2a_2^3}{27} - \frac{a_1 a_2}{3} + a_0 = -\frac{-34}{27}.$$

Now we obtain

$$P = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} = \sqrt[3]{\frac{17}{27} + \sqrt{\frac{8}{729} + \frac{289}{729}}} = \sqrt[3]{\frac{17}{27} + \frac{3}{27}\sqrt{33}} = \frac{1}{3}\sqrt[3]{17 + 3\sqrt{33}},$$

and

$$Q = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} = \frac{1}{3}\sqrt[3]{17 - 3\sqrt{33}}.$$

Let $\omega = 1^{1/3}$ be a primitive cube root of unity. Then the poles of $|T|$ are:

$$\begin{aligned}\alpha &= \tfrac{1}{3}\sqrt[3]{17 + 3\sqrt{33}} + \tfrac{1}{3}\sqrt[3]{17 - 3\sqrt{33}} - \tfrac{1}{3}\\ \beta &= \tfrac{\omega}{3}\sqrt[3]{17 + 3\sqrt{33}} + \tfrac{\omega^2}{3}\sqrt[3]{17 - 3\sqrt{33}} - \tfrac{1}{3}\\ \gamma &= \tfrac{\omega^2}{3}\sqrt[3]{17 + 3\sqrt{33}} + \tfrac{\omega}{3}\sqrt[3]{17 - 3\sqrt{33}} - \tfrac{1}{3}\end{aligned}$$

Checking the moduli of these numbers,

$$\begin{aligned}|\alpha| &\approx 0.5436890126920764\\ |\beta| &\approx 1.3562030656262953\\ |\gamma| &\approx 1.3562030656262953\end{aligned}$$

we see that $\alpha$ is closest to the origin.

6. The Souped-Up Hadamard Theorem says: If $f(z) = \sum a_n z^n$ is analytic in a disc of radius $R$ about 0, except for a simple pole at distance $r < R$ from 0, then $|a_n| \sim c(\frac{1}{r})^n, c > 0$. Let
$$\tau = \tfrac{1}{|\alpha|}.$$
Then for $|\alpha| < R < |\beta| = |\gamma|$, the theorem gives that
$$t_n \sim c\tau^n.$$

7. We use the characteristic equation to solve the recurrence relation $t_n = t_{n-3} + t_{n-2} + t_{n-1}$ with the initial conditions $t_0 = 1, t_1 = 1, t_2 = 2$ and obtain
$$t_n = \frac{\beta\gamma - \beta - \gamma + 2}{(\beta - \alpha)(\gamma - \alpha)}\alpha^n + \frac{\alpha\gamma - \alpha - \gamma + 2}{(\beta - \alpha)(\beta - \gamma)}\beta^n + \frac{\alpha\beta - \alpha - \beta + 2}{(\beta - \gamma)(\gamma - \alpha)}\gamma^n$$
With this, we compute $t_{100} = 180396380815100901214157639$.
Also, using Mathematica, we obtain $\tau^{100} = 2.9170531916003307 \times 10^{26}$. Since
$$t_n \sim c\tau^n \implies \frac{t_n}{\tau^n} \sim c,$$
we obtain $c \approx 0.6184199223193912$.

8. Let $b_2^{(k)}$ be the $n^{\text{th}}$ $k$-bonacci number. Then the isomorphism
$$B^{(k)} \cong Z^k B^{(k)} + Z^{k-1} B^{(k)} + \ldots + Z B^{(k)} + 1$$
can be decategorified to yield
$$\begin{aligned} |B^{(k)}| = \sum_{n\geq 0} b_n^{(k)} z^n &= z^k |B^{(k)}| + z^{k-1}|B^{(k)}| + \ldots + z|B^{(k)}| + 1 \\ &= \sum_{n\geq 0} t_n z^{n+k} + \sum_{n\geq 0} t_n z^{n+k-1} + \ldots + \sum_{n\geq 0} t_n z^n + 1 \\ &= \sum_{n\geq k} t_n z^n + \sum_{n\geq k-1} t_n z^n + \ldots + \sum_{n\geq 1} t_n z^n + 1 \end{aligned}$$

For initial conditions, we still have $b_0^{(k)} = 1, b_1^{(k)} = 1, b_2^{(k)} = 2$, and (for $k > 2$) $b_3^{(k)} = 4$. But (for $k > 3$) $b_4^{(k)}$ takes the 7 partitions of $t_4$ and adds one more: $\bullet\bullet\bullet\bullet$ . So $b_4^{(k)} = 8$.

In fact, $\forall k, b_0^{(k)} = 0$ and $b_n^{(k)} = 2^{n-1}$ for $n \leq k$. Reason: for the first $k$ terms, we can chop into blocks of $1, 2, 3, \ldots, k$. This means that there are *no* restrictions on where to place the partitioning lines, so we have a simple counting problem:
$$2 \text{ options for each of } n-1 \text{ choices} \implies 2^{n-1} \text{ ways.}$$

9. The $n^{\text{th}}$ $k$-bonacci numbers:

| | n=0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| k=1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |
| 3 | 1 | 1 | 2 | 4 | 7 | 13 | 24 | 44 | 81 | 149 | 274 |
| 4 | 1 | 1 | 2 | 4 | 8 | 15 | 29 | 56 | 108 | 208 | 401 |
| 5 | 1 | 1 | 2 | 4 | 8 | 16 | 31 | 61 | 120 | 236 | 464 |
| 6 | 1 | 1 | 2 | 4 | 8 | 16 | 32 | 63 | 125 | 248 | 492 |
| 7 | 1 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 127 | 253 | 504 |
| 8 | 1 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 255 | 509 |
| 9 | 1 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 511 |
| 10 | 1 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |

10. Define the $\infty$-bonacci number to be the number of ways of chopping a totally ordered $n$-element set into blocks of arbitrarily long integer length.

As mentioned in 8, $b_n^{(k)} = 2^{n-1}$ for $n \leq k$. Thus, as $k \to \infty$, $b_n^{(k)} = 2^{n-1}$ for more and more of the initial part of the sequence. In fact, if we define a metric $\sigma$ on the space of sequences by

$$\sigma(u, v) = r^{\min\{j \,:\, u_j \neq v_j\}}$$

for some fixed $0 < r < 1$ (say $r = \frac{1}{2}$), then

$$\lim_{k\to\infty} \{b_n^{(k)}\} = \{2^{n-1}\},$$

except for $n = 0$, because $b_0^{(k)} = 1, \forall k$. I.e., $b_n^{(\infty)} = 2^{n-1}$. Thus the generating function will be

$$|B^{(k)}| = \sum_{n\geq 1} 2^{n-1} z^n + 1$$

11. The $n^{\text{th}}$ $\infty$-bonacci number is $2^{n-1}$ (but 1 when $n = 0$), by the above.