# Math 260: Tribonacci, $k$-bonacci, $\infty$-bonacci

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**1.** *Recursive definition of $T$.*

A $T$-structure on $S$ is either $S$ being the empty set, or a block of length 1, 2, or 3 followed by a $T$-structure on the rest. It follows that

$$T \simeq 1 + (Z + Z^2 + Z^3) \times T.$$

**2.** *Generating function (implicit).*

The decategorification of the preceding isomorphism is

$$|T|(z) = 1 + (z + z^2 + z^3)|T|(z).$$

**3.** *Recurrence relation.*

Let $|T|(z) = \sum_{n\geq 0} t_n z^n$. Then,

$$\sum_{n\geq 0} t_n z^n = 1 + \sum_{n\geq 0} t_n z^{n+1} + \sum_{n\geq 0} t_n z^{n+2} + \sum_{n\geq 0} t_n z^{n+3}$$

so

$$\sum_{n\geq 0} t_n z^n = 1 + \sum_{n\geq 1} t_{n-1} z^n + \sum_{n\geq 2} t_{n-2} z^n + \sum_{n\geq 3} t_{n-3} z^n$$

and

$$\begin{cases} t_0 = 1 \\ t_1 = t_0 \\ t_2 = t_1 + t_0 \\ t_n = t_{n-1} + t_{n-2} + t_{n-3} \quad n \geq 3 \end{cases}$$

**4.** *Generating function (explicit).*

The equation in part 2 can be solved for $|T|(z)$ to obtain $|T(z)| = (1 - z - z^2 - z^3)^{-1}$.

**5.** *The closest pole to the origin.*

We need to find the root of $z^3 + z^2 + z - 1 = 0$ closest to the origin. To eliminate the $z^2$ term, we let $z = w - 1/3$, obtaining $w^3 + 2w/3 - 34/27 = 0$. By Descartes' sign rule, there is at most one positive root $w_0$ and no negative roots. To find the unique real root, we need to put a hyperbolic spin on Viete's trigonometric method. Observe that $4\sinh^3\eta + 3\sinh\eta = \sinh 3\eta$. Letting $w = \frac{2\sqrt{2}}{3}\sinh\eta$, we get $\sinh 3\eta = \frac{17}{2\sqrt{2}}$. Call this number $a$. Then, $e^{3\eta} = a + \sqrt{1+a^2} = \frac{3\sqrt{33}+17}{2\sqrt{2}}$. In other words,

$$z_0 = \frac{\sqrt{2}}{3}\left[\frac{\sqrt[3]{3\sqrt{33}+17}}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt[3]{3\sqrt{33}+17}}\right] - \frac{1}{3} \approx 0.54369$$

This is the root closest to the origin because the product of the three roots is the negative of the independent term of the polynomial, namely 1, and the two complex roots must then have absolute value $z_0^{-1/2} \approx 1.3562$. We can factor $z^3 + z^2 + z - 1$ as

$$(z - z_0)\big(z^2 + (1+z_0)z + (1 + z_0 + z_0^2)\big) = (z - z_0)\left[\left(z + \frac{1+z_0}{2}\right)^2 + \frac{3 + 2z_0 + 3z_0^2}{4}\right],$$

so the complex roots are

$$z_\pm = \frac{-(1+z_0) \pm i\sqrt{3 + 2z_0 + 3z_0^2}}{2} \approx -0.77184 \pm 1.1151i.$$

**6,7.** *Asymptotics.*

We have

$$\frac{1}{z^3+z^2+z-1} = \frac{1}{(z-z_0)\big(z^2+(1+z_0)z+(1+z_0+z_0^2)\big)} = \frac{A}{z-z_0} + \frac{Bz+C}{z^2+(1+z_0)z+(1+z_0+z_0^2)}$$

where

$$\begin{array}{rcccccl} & A & + & B & & & = 0 \\ (1+z_0) & A & -z_0 & B & + & C & = 0 \\ (1+z_0+z_0^2) & A & & & -z_0 & C & = 1 \end{array}$$

We have $1+2z_0+3z_0^2A = 1$. The term $\frac{A}{z-z_0}$ dominates the asymptotic behaviour of the coefficients of the power series expansion because the complex roots are larger in absolute value than $z_0$, so

$$|T(z)| \sim \frac{-1}{1+2z_0+3z_0^2}\frac{1}{z-z_0} = \frac{1}{z_0+2z_0^2+3z_0^3}\frac{1}{1-(z/z_0)} = \frac{1}{z_0+2z_0^2+3z_0^3}\sum_{n\geq 0}(z/z_0)^n.$$

Hence, $\tau = z_0^{-1} \approx 1.8393$ and $c = (z_0+2z_0^2+3z_0^3)^{-1} \approx 0.61842$

The following `Octave` code

```
a=zeros[0:100];
a(1)=1; a(2)=1; a(3)=2;
for i=4:101
  a(i)=a(i-1)+a(i-2)+a(i-3);
endfor;
a[101]
```

finds $t_{100} \approx 1.8040 \cdot 10^{26}$, while $z_0^{-100} \approx 2.9171 \cdot 10^{26}$. The ratio of the two is, indeed, $c \approx 0.61842$.

**8,9.** *k-bonacci.*

If $K$ is the $k$-bonacci structure type,

$$K \simeq 1 + (Z + Z^2 + \cdots + Z^k) \times K$$

so

$$|K|(z) = \frac{1}{1-z-z^2-\cdots-z^k}.$$

The recurrence relation for the $n$th $k$-bonacci number, $k_n$ is

$$\begin{cases} t_n = 0 & n < 0 \\ t_0 = 1 & \\ t_n = t_{n-1} + t_{n-2} + \cdots + t_{n-k} & n \geq 1 \end{cases}$$

This recurrence relation produces the following numbers:

| | n=0 | n=1 | n=2 | n=3 | n=4 | n=5 |
|---|---|---|---|---|---|---|
| k=1 | 1 | 1 | 1 | 1 | 1 | 1 |
| k=2 | 1 | 1 | 2 | 3 | 5 | 8 |
| k=3 | 1 | 1 | 2 | 4 | 7 | 13 |
| k=4 | 1 | 1 | 2 | 4 | 8 | 15 |
| k=5 | 1 | 1 | 2 | 4 | 8 | 16 |

**10, 11.** *$\infty$-bonacci.*

Using the fact that $\frac{1}{1-z} = 1 + z + \cdots + z^n + \cdots$, we can write $z + z^2 + \cdots + z^n + \cdots = \frac{z}{1-z}$, and so the generating function of the $\infty$-bonacci numbers satisfies

$$|I|(z) = \frac{1}{1-\frac{z}{1-z}} = \frac{1-z}{1-2z}.$$

We can write this as

$$|I|(z) = (1-z)\sum_{n\geq 0}(2z)^n = \sum_{n\geq 0}(2z)^n - \sum_{n\geq 0}2^n z^{n+1} = 1 + \sum_{n\geq 1}2^n z^n - \sum_{n\geq 1}2^{n-1}z^n = 1 + \sum_{n\geq 1}2^{n-1}z^n.$$

So the $n$-th $\infty$-bonacci number is 1 when $n=0$, and $2^{n-1}$ for $n \geq 1$. This is because an $\infty$-bonacci structure is an arbitrary partition of an ordered set, and for $n \geq 1$, an $n$-element set has $n-1$ possible positions where a block boundary can be present or absent.