QUANTUM GRAVITY SEMINAR

Winter 2004

6 Jan 2004

Structure Types

We've defined a structure type to be a functor

$$F: \mathrm{FinSet}_0 \to \mathrm{Set}$$

We think of F as assigning to any finite set S the "set of all F-structures on S, F_S.

Given a structure type F, we can define its <u>generating function</u> $|F| \in \mathbb{C}[[z]]$, by

$$|F|(z) = \sum_{n \geq 0} \frac{|F_n|}{n!} z^n$$

where F_n is the set of F-strs. on the n-elt set, "n", & $|F_n|$ is its cardinality.

We saw that these give new insight into the quantum harmonic oscillator. This quarter, we'll

1) Generalize to formal power series with many variables (possibly infinitely many) which are generating fns of "n-sorted str. types" $F: \mathrm{FinSet}_0{}^n \to \mathrm{Set}$.

This lets us categorify the quantum harmonic oscillator with n degrees of freedom, especially $n = \infty$. Quantum field theory is about oscillators w. infinitely many degrees of freedom:

$n = 1$

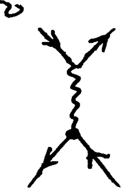
pendulum

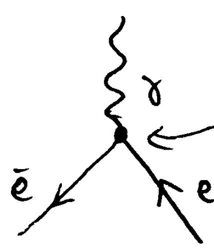
Config space is $\mathbb{R}$

$n = \infty$

violin string

config space is infinite-dimensional, "$\mathbb{R}^\infty$".

Here creation and annihilation operators create "quanta" ← particles!
of different sorts... whereas last quarter, in the $n=1$ case, we only had <u>one</u> creation operator & <u>one</u> annihilation operator. We'll be able to categorify the theory of Feynman diagrams:  ↑time



$a_\gamma^* \, a_e \, a_{\bar{e}}$

2) Understand more deeply how structure types are "categorified formal power series." We've seen hom(FinSet$_0$, Set) is a category analogous to the set $\mathbb{C}[\![z]\!]$. We need to make this analogy precise.

To do this, we'll need to generalize structure types to "stuff types." We'll learn about: properties, structure, 'n' stuff! We need this to categorify Feynman diagrams

3) Have Fun: use structure types to do all sorts of combinatorics, number theory, ...

Example of Fun:

Recall:

1) If F is "being a totally ordered n-elt. set",

$$|F|(z) = z^n$$

since $|F_n| = n!$ & $|F_m| = 0$ if $m \neq n$.

So we call F "Z^n":

$$|Z^n| = z^n.$$

2) If F & G are str. types, to put an $F+G$-str on S we: choose either F or G & then put such a structure on S.

$$|F+G| = |F| + |G|$$

3) To put an FG-str. on S we chop S into 2 subsets & put an F str. on first part, a G-str. on second part.

$$|FG| = |F||G|.$$

Now: let an F-str. on S be a way of totally ordering it & then chopping it up into "blocks" of size 1 or 2:

$$7\ 1 \mid 2 \mid 4\ 3 \mid 5 \mid 6$$

is an F-str. on $\{1, \ldots, 7\}$

A subset $T \subseteq S$ is a <u>block</u> if $x, y \in T$, $x \leq z \leq y$ $\Rightarrow z \in T$

$$|F_0| = 1$$

$$|F_1| = 1$$

$$|F_2| = 4$$

$xy \quad yx$

$x|y \quad y|x$

<u>Note:</u>

$$F \cong Z^2 F + ZF + 1$$

i.e. to put an F str. on S, either chop it in two & put the structure of being a totally ordered 2-elt set ~~on the first part~~ or being a totally ordered 1-elt set on the first part, and then put an F str. on the second part. (or $F = \emptyset$)

So:

$$|F| = z^2|F| + z|F| + 1$$

$$(1 - z - z^2)|F| = 1$$

$$|F|(z) = \frac{1}{1 - z - z^2}$$

Note:

$$|F|(z) = 1 + (z + z^2) + (z + z^2)^2 + (z + z^2)^3 + \cdots$$

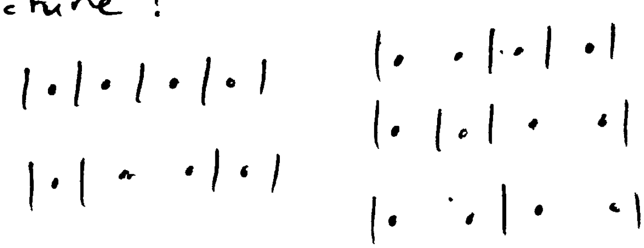
$$\begin{aligned} = \quad & 1 \, + \\ & z + z^2 + \\ & z^2 + 2z^3 + z^4 + \\ & z^3 + 3z^4 + 3z^5 + z^6 + \\ & z^4 + 4z^5 + 6z^6 + 4z^7 + z^8 + \cdots \end{aligned}$$

$$= 1 + 1z + 2z^2 + 3z^3 + 5z^4 + \cdots$$

$$= 1 + z + \frac{4z^2}{2!} + \frac{18z^3}{3!} + \cdots$$

Note: $|F_n| = f_n \cdot n!$, where $n!$ comes from the choice of orderings & f_n comes from choice of block structure:

$f_4 = 5$



Since $|F| = \sum_{n \geq 0} f_n z^n$,

$$|F| = z^2|F| + z|F| + 1$$

says $\sum_{n\geq 0} f_n z^n = \sum_{n \geq 0} f_n z^{n+1} + f_n z^{n+2} + 1$

$$\sum_{n\geq 0} f_n z^n = \sum_{n\geq 1} f_{n-1} z^n + \sum_{n \geq 2} f_{n-2} z^n + 1$$

so

$$f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2$$

& we know $f_0 = f_1 = 1$, so f_n is the nth Fibonacci number!

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8 Jan 2004

(Lots of people gone to MAA/AMS meeting today...)

We saw that if F is the structure type "total ordering & partition into blocks of length 1 or 2, then

$$F \cong Z^2 F + ZF + 1$$

$$\text{so } |F| = \frac{1}{1-z-z^2}$$

We can study asymptotics of $|F_n| = n! f_n$ by studying radius of convergence, poles, etc. of $|F|(z)$. This is quite general, so let Φ

be any str. type & $|\Phi|(z) = \sum_{n\geq 0} \frac{|\Phi_n|}{n!} z^n = \sum_{n\geq 0} a_n z^n$ (where $a_n = f_n$ in our example). Hadamard's Theorem says that if R is the radius of convergence of $\sum_{n\geq 0} a_n z^n$ then $\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$

This implies that for any $\varepsilon > 0$, eventually

$$\sqrt[n]{|a_n|} \leq \frac{1}{R} + \varepsilon$$

or

$$|a_n| \leq \left(\frac{1}{R} + \varepsilon\right)^n$$

We don't get any good lower bounds unless $\lim \sqrt[n]{|a_n|}$ exists, in which case $\forall \varepsilon > 0$ we eventually have

$$|a_n| \geq \left(\frac{1}{R} - \varepsilon\right)^n$$

In our example:

$$\frac{1}{1-z-z^2}$$

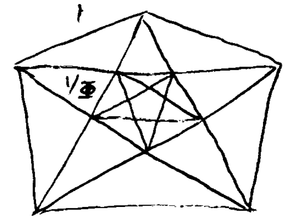
has simple poles at roots of $z^2 + z - 1 = 0$

$$z = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

Golden ratio $\frac{\sqrt{5}+1}{2} = \Phi = 1.6180339\ldots$

$\frac{\sqrt{5}-1}{2} = \Phi^{-1} = 0.6180339\ldots$

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}$$



$$= \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}$$

<u>So:</u> the radius of convergence of $\sum\limits_{n \geq 0} f_n z^n$ is

$$\left(\text{distance of closest pole from origin of } \frac{1}{1-z-z^2}\right) = \frac{1}{\Phi}$$

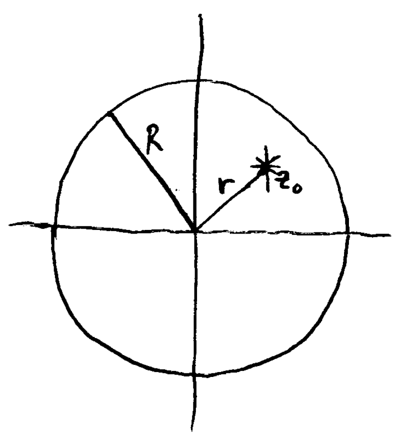
So, Hadamard's Theorem says

$$|f_n| \leq (\Phi + \varepsilon)^n$$

for large enough n.

<u>Souped-up Hadamard's Theorem:</u>

Suppose $f(z) = \sum a_n z^n$ is analytic in a disc of radius R about the origin except for a simple pole at distance $r < R$ from the origin:



Then
$$|a_n| \sim c\left(\frac{1}{r}\right)^n \qquad c > 0$$
"asymptotic"

i.e.
$$\lim_{n\to\infty} \frac{|a_n|}{c\left(\frac{1}{r}\right)^n} = 1$$

<u>Proof:</u> Can find $k \neq 0$ s.t.
$$f(z) = \tilde{f}(z) + \frac{k}{z - z_0}$$
where f is analytic on the disc of radius R.

Let
$$\tilde{f}(z) = \sum \tilde{a}_n z^n$$
so
$$\sum a_n z^n = \sum \tilde{a}_n z^n - \frac{k}{z_0}\sum\left(\frac{z}{z_0}\right)^n$$

$$\frac{k}{z-z_0} = \frac{-k/z_0}{1 - z/z_0} = \frac{-k}{z_0}\sum\left(\frac{z}{z_0}\right)^n$$

By Hadamard's Thm,
$$|\tilde{a}_n| \leq \left(\frac{1}{R} + \varepsilon\right)^n \text{, eventually,}$$
but the other term has:
$$\left| -\frac{k}{z_0}\left(\frac{z}{z_0}\right)^n \right| = c\left(\frac{1}{r}\right)^n$$

Choose ε s.t. $\frac{1}{r} > \frac{1}{R} + \varepsilon$ and conclude $|\tilde{a}_n|$ is dwarfed by $c\left(\frac{1}{r}\right)^n$ for large n, so $|a_n| \sim c\left(\frac{1}{r}\right)^n$. ∎

So in our example we actually find

$$f_n \sim c\,\Phi^n.$$

We can do much better, though.

$$\frac{1}{1-z-z^2} = \frac{A}{z+\Phi} + \frac{B}{z-\Phi^{-1}}$$

$$= \frac{A/\Phi}{1+z/\Phi} - \frac{B\Phi}{1-z\Phi}$$

$$= \frac{A}{\Phi}\sum\left(\frac{-z}{\Phi}\right)^n - B\Phi\sum(z\Phi)^n = \sum f_n z^n$$

~~In fact~~:

So: $$f_n = \frac{A}{\Phi}\left(-\frac{1}{\Phi}\right)^n - B\Phi(\Phi)^n$$

In fact

$$f_n = \frac{\Phi^{n+1} - (-\Phi)^{-(n+1)}}{\sqrt{5}}$$

Since $\Phi^{-(n+1)} \longrightarrow$ as $n \to \infty$, this means that as $n \to +\infty$, $\left|f_n - \frac{\Phi^{n+1}}{\sqrt{5}}\right| \to 0$. I.e. $\frac{\Phi^n}{\sqrt{5}}$ gets closer and closer to an integer! By the way, $\exists\, a > 0$ s.t. $a \notin \mathbb{Q}$ but a^n gets closer & closer to integers as $n \to \infty$.

In fact:

$$\frac{\Phi^1}{\sqrt{5}} = .73\ldots \text{ has } 1 = f_0 \text{ as its closest integer}$$

$$\frac{\Phi^2}{\sqrt{5}} = 1.13\ldots \text{ has } 1 = f_1 \quad \text{"} \qquad \text{"}$$

$$\frac{\Phi^3}{\sqrt{5}} = 1.89\ldots \text{ has } 2 = f_2 \quad \text{"} \qquad \text{"}$$

Therefore

closest integer

$$f_n = \left[\frac{\Phi^n}{\sqrt{5}} \right]$$

<u>Moral:</u> The growth rate of $|\Phi_n|$ for some str. type Φ is controlled by singularities (poles, branch cuts, ...) of $|\Phi|(z)$. If $|\Phi|(z)$ is a rational function, we can use partial fractions to completely solve for $|\Phi_n|$ in "closed form."

So: there's a classification of str. types according to their generating fns. — e.g. "rational structure types" have rational generating functions: this gives a full subcategory of the category of str. types, $\hom(\mathrm{finSet}_0, \mathrm{Set})$, and this subcategory is closed under $+$ & $\cdot$.