Last time we had:

\[ \text{Set} \xrightarrow{S \rightarrow CS} \text{Vect}_C \]
\[ \quad \downarrow V \mapsto SV \]
\[ \quad \text{CommAlg}_C \]

*Given a set \( S \) of "vibrational modes" we can form the complex vector space \( CS \) with basis \( S \); it's just the classical phase space. E.g. if \( S = 1 \), \( CS = C \) is the classical phase space for the harmonic oscillator with 1 degree of freedom:

\( C \ni z = q + ip \)

describes position & momentum of the oscillator. If \( S = n \), \( CS = C^n \) & we get \( (z_1, \ldots, z_n) = (q_1 + ip_1, \ldots, q_n + ip_n) \).

Given a phase space \( V \subseteq \text{Vect}_C \), we can form the symmetric tensor algebra on it, \( SV \); this is the "pre-Fock space", whose Hilbert space completion is the Hilbert space of the quantum harmonic oscillator. E.g. if \( V = C \), \( SV = C[z] \). If \( V = C^n \), \( SV = C[z_1, \ldots, z_n] \).
There's another route:

\[
\begin{array}{cc}
\text{Set} & \xrightarrow{s \mapsto \mathbb{C}s} & \text{Vec}_\mathbb{C} \\
\downarrow \quad \alpha & \uparrow \quad \text{V} \mapsto \text{SV} & \text{or } "\text{V} \mapsto \text{[V]}" \quad ?.
\end{array}
\]

\[
\begin{array}{c}
\alpha \\
\downarrow
\end{array}
\]

\[
\begin{array}{c}
\text{CommMon} \\
\xrightarrow{M \mapsto \text{CM}} & \text{CommAlg}_\mathbb{C}
\end{array}
\]

Here \([S]\) is the free commutative monoid on \(S\): e.g. if \(S = \{z_1, \ldots, z_n\}\) then \([S]\) is the comm. monoid of monomials \(z_1^{p_1} \cdots z_n^{p_n}\). Given a comm. monoid \(M\), \(\text{CM}\) consists of all formal linear combinations of elts. of \(M\), & this is a commutative algebra. This square commutes up to natural isomorphism:

\[
\alpha_s : \mathbb{C}[S] \rightarrow [S].
\]

i.e. "\(\mathbb{C}\) commutes with \([\cdot]\)."

E.g.

\[
\alpha_s^{-1}((z + 2i z_2)(z_3 - 7z_4)) = z_3 - 7z_1 z_4 + 2i z_2 z_3 - 14i z_3 z_4.
\]

\(\alpha_s^{-1}\) is just the distributive law in action! (it lets us write products of sums as a sum of products)

So, whenever we have a commutative square of free functors, people call it a distributive law.
Physically, if $S$ is a set of "vibrational modes" or "types of particle", the free comm monoid $[S]$ is the set of "collections of particles" - e.g. $z_1^p_1 \ldots z_n^p_n$ represents a collection with $p_i$ particles of type $i$. These monomials form a basis for the polynomials, so physically these states form a basis of Fock space.

We would like to generalize this in a way that is not specific to $C$. Let's replace $C$ by any commutative rig $R$:

\[
\begin{array}{ccc}
\text{Set} & \xrightarrow{S \mapsto RS} & \text{R-Mod} \\
\downarrow & & \downarrow \\
\text{CommMon} & \xrightarrow{M \mapsto RM} & \text{Comm R-Alg}
\end{array}
\]

$R$-modules & commutative $R$-algebras are defined for comm. rigs just as for comm. rings.

(Note: if your algebra book defines an $R$-module as an abelian group, replace group by monoid - the negatives are superfluous even in the ring case.)

Again, the diagram commutes up to natural isomorphism.
Let's try $R = \mathbb{N}$, the free comm. rig on no generators:

$\text{Set} \xrightarrow{\text{taking formal sums}} \text{Comm Mon} (= \mathbb{N}/\text{Mod}) \xrightarrow{\text{taking formal products}} \text{Comm Rig} (= \text{Comm } \mathbb{N}/\text{Alg})$

Another famous example: $R = \mathbb{Z}$, the free commutative ring on no generators:

$\text{Set} \xrightarrow{\text{taking formal sums \& differences}} \text{AbGrp} \xrightarrow{\text{taking formal products}} \text{Comm Mon} \xrightarrow{\text{taking formal sums \& differences}} \text{Comm Ring}$
Let's do an example.

\[ \sin x, \sin 2x, \sin 3x, \ldots \Rightarrow \mathbb{N}^+ \]

A violin string has vibrational modes forming the set \( \mathbb{N}^+ \) if our string has length \( \pi \). The frequency of the vibration of the \( n \)th mode is proportional to \( n \), \( n \) depending on tension, mass density of the string.

Let's just say \( \nu_n = n \) for simplicity. In QM we learn that energy \( \nu \cdot \text{frequency} \) so we can also say \( E_n = n \).

We'll count the number of states having energy \( E \) for the quantized string.