

10 February 2004

## Generating Functions from Partition Functions.

We saw that a generating function

$$\sum_{n \in \mathbb{N}} p(n) z^n$$

where  $p(n)$  is the number of states of energy  $n$  for some physical system, is secretly the partition function

$$Z(\beta) = \sum_{n \in \mathbb{N}} p(n) e^{-\beta n} \quad \beta = \frac{1}{kT}$$

if we let  $z = e^{-\beta}$ .

Both generating functions of str. types & partition functions (of classical & quantum thermodynamic systems) are generalizations of cardinality:

$$\text{FinSet} \xleftarrow{S \mapsto (S, \text{counting meas.}, H=0)} \text{CTherm}$$

This is a 2-rig homomorphism where CTherm is a 2-rig with objects:

$$\mathcal{X} = (X, dx, H)$$

measurable space

measure on  $X$

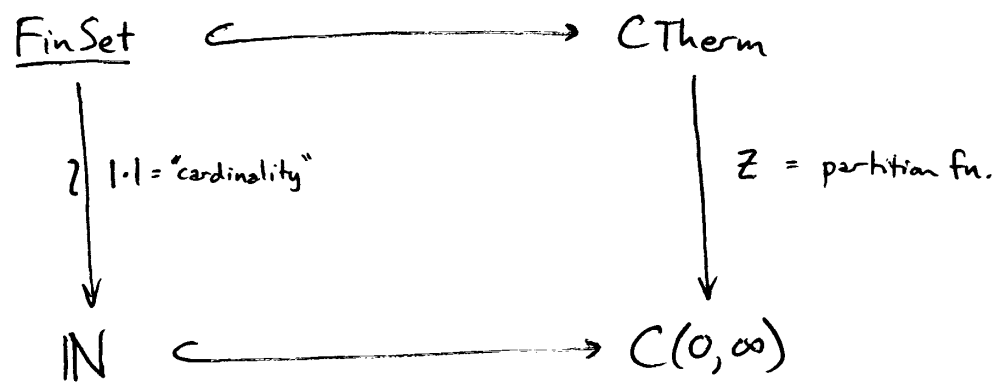
$H: X \rightarrow \mathbb{R}$  measurable fn.

such that

$$Z(\mathcal{X})(\beta) = \int_X e^{-\beta H(x)} dx$$

converges  $\forall \beta \in (0, \infty)$

So, decategorifying we get rig homomorphisms:

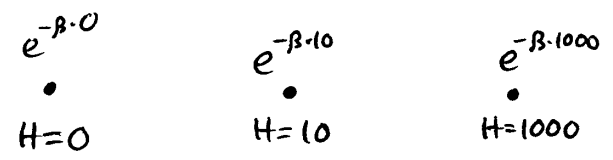


So: "A partition function is like temperature-dependent cardinality"

E.g.: if  $X$  has counting measure but  $H: X \rightarrow \mathbb{R}$  is nonnegative, then

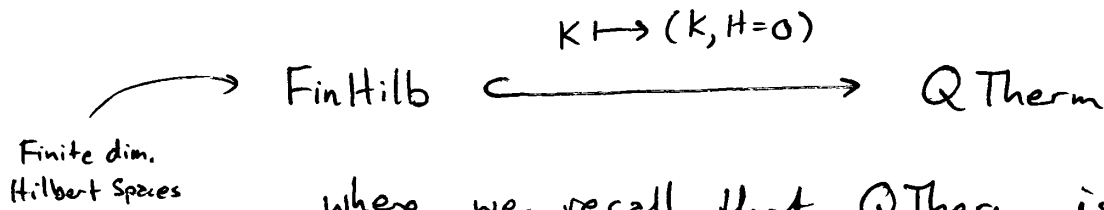
$$Z(X)(\beta) = \sum_{x \in X} e^{-\beta H(x)}$$

This is just the cardinality of  $X$  when  $\beta = 0$  i.e. when  $T$  is infinite. But when the system is cold, high-energy states get low weights  $e^{-\beta H(x)}$ :



↖ This element scarcely counts until  $\beta$  is very small ( $\sim \frac{1}{1000}$ ) i.e. until the system is very hot.

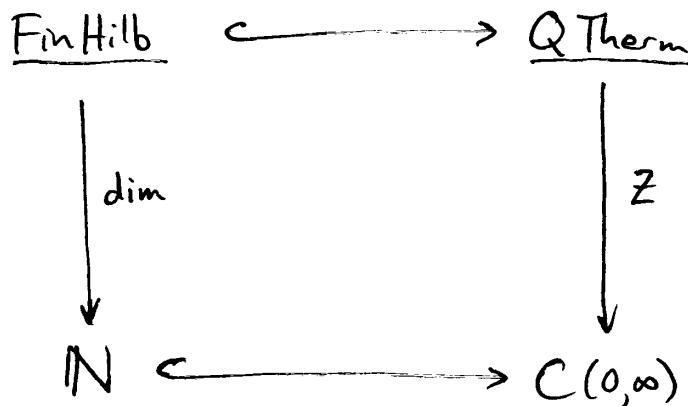
Similarly for quantum thermodynamic systems, there's a 2-rig homomorphism:



where we recall that QTherm is a 2-rig with objects

$$\begin{array}{l}
 \mathcal{X}(K, H) \\
 \left( \begin{array}{l} \text{Hilbert} \\ \text{space} \end{array} \right) \left( \begin{array}{l} \text{self-adjoint op. on } K \\ \text{s.t. } \text{tr}(e^{-\beta H}) = Z(\chi)(\beta) \\ \text{converges } \forall \beta \in (0, \infty) \\ \text{(note: } H=0 \Rightarrow \text{tr}(e^{-\beta H}) = \text{dim } K.) \end{array} \right)
 \end{array}$$

Here we get a commutative diagram of rig homomorphisms:



So in the quantum case the partition function is like temperature-dependent dimension.

Generating functions of structure types are similar:

$$k \mapsto k = \left( \begin{array}{l} \text{str. of "being } \emptyset \text{ and choosing"} \\ \text{a number } \in \{1, 2, \dots, k\} \end{array} \right)$$

$$\text{FinSet} \hookrightarrow [\text{Structure Types}]$$

is a 2-rig homo &

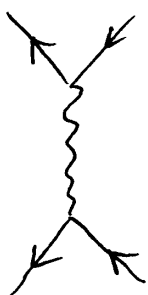
$$\underline{\text{FinSet}} \hookrightarrow |[\text{Structure Types}]|$$

$$\begin{array}{ccc} \text{cardinality } | \cdot | & & \text{generating function} \\ \downarrow & & \downarrow \\ \mathbb{N} & \hookrightarrow & \mathbb{Q}^+[z] \end{array}$$

This is why generating functions can sometimes be partition functions after we make the substitution

$$z = e^{-\beta}$$

Our next goal in the seminar will be to generalize structure types to "stuff types" & use these to categorify the theory of Feynman diagrams:



Let's start with an innocent-looking puzzle:

Given a structure type  $F$ , we can decategorify & get  $|F|$ , & then evaluate it at  $z = z_0$  to get

$$|F|(z_0) \text{ - a number.}$$

(if the series converges, anyway). Can we get the same result by first evaluating  $F$  at something  $\leftarrow$  some  $Z_0$   $\leftarrow$  & then decategorifying the result,  $F(Z_0)$  (to get the same number). I.e. we'd like this to make sense:

$$|F(Z_0)| = |F|(|Z_0|)$$

But: what sort of thing is  $Z_0$ ? What sort of thing is  $F(Z_0)$ ? & how do we compute numbers  $|Z_0|$  &  $|F(Z_0)|$ ?

Let's suppose for starters that  $Z_0$  is a finite set. Then  $|Z_0|$  is just its cardinality. Look at  $F(Z_0)$  vs.  $|F(|Z_0|)|$ :

$$|F(|Z_0|)| = \sum_{n \in \mathbb{N}} \frac{|F_n| |Z_0|^n}{n!}$$

where  $F_n$  is the set of  $F$ -structures on  $n$ . So we want to make sense of

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n Z_0^n}{n!}$$

The question becomes: what sort of thing is  $\frac{F_n Z_0^n}{n!}$  & how can we get  $|\frac{F_n Z_0^n}{n!}| = \frac{|F_n| |Z_0|^n}{n!}$  since then we might have  $|F(Z_0)| = |F(|Z_0|)|$ . (we would still have to make sense of the sum)

Note:  $F_n$  is a set;  $Z_0^n$  is a set, but we can't find a set  $\frac{F_n Z_0^n}{n!}$  since then  $|\frac{F_n Z_0^n}{n!}| \in \mathbb{N}$  while  $\frac{|F_n| |Z_0|^n}{n!} \notin \mathbb{N}$  in general. So  $\frac{F_n Z_0^n}{n!}$

shouldn't be a set. What is  $n!$ ? It's really the permutation group  $n!$ . So we seek a recipe for taking a set  $S$ , a group  $G$  (acting on  $S$ ) & get something " $S/G$ " that's

not a set & can have fractional cardinality,  
namely  $|S|/|G|$ .

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12 February 2004

## GROUPOID CARDINALITY

- groupoids as "fractional sets."

If  $F$  is a structure type &  $Z_0$  is a finite set, we can evaluate  $|F|$  at  $Z_0$ :

$$|F|(Z_0) = \sum_{n \geq 0} \frac{|F_n|}{n!} |Z_0|^n$$

Can we evaluate  $F$  at  $Z_0$ ?

$$F(Z_0) = \sum_{n \geq 0} \frac{F_n \times Z_0^n}{n!}$$

& get something " $F(Z_0)$ " with

$$|F(Z_0)| = |F|(Z_0) ?$$

We need to be able to pass  $|\cdot|$  through sums and quotients. The second part of this is the hard part: given a set  $S$  & a group  $G$  acting on  $S$  (here  $G = n!$ , the permutation

group on  $n$ ), can we define some gizmo  
 "S/G" with "cardinality"  $|S/G| = |S|/|G|$ ?

One obvious (but alas, wrong) guess: Let S/G  
 be the set of orbits of the action of G  
 on S:

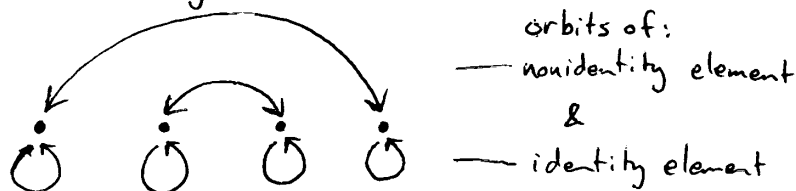
$$[s] = \{sg : g \in G\}$$

(note: right-action)

Sometimes this works:

if  $S = \cdot \cdot \cdot \cdot$

&  $G = \mathbb{Z}_2$  acting as left-right reflection:



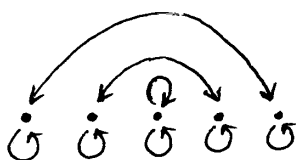
So

$$|S/G| = 2 = |S|/|G|$$

But sometimes it doesn't work:

if  $S = \cdot \cdot \cdot \cdot \cdot$ ,  $G = \mathbb{Z}_2$  with reflection action

we get





Now not all orbits are the same size!

$$|S/G| = 3 \neq 2\frac{1}{2} = |S|/|G|$$

The problem: the dot in the middle should count as "half a dot" to get things to work out! This dot is "half a dot" because its stabilizer has two elements:

$$\text{Stab}(s) = \{g \in G : sg = s\}.$$

Note: Big stabilizer  $\iff$  small orbit:

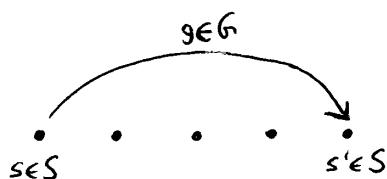
$$|[s]| = \frac{|G|}{|\text{Stab}(s)|}$$

The Solution: given a group  $G$  acting on a set  $S$ , form the weak quotient  $S//G$ , which is the groupoid whose objects are elements of  $S$  & whose morphisms are pairs  $(s, g) \in S \times G$ , thought of as morphisms:

$$g: s \longrightarrow s'$$

where  $s' = sg$ .

e.g.



We compose morphisms via:

$$g: s \rightarrow s' \quad \& \quad h: s' \rightarrow s'' \implies gh: s \rightarrow s''$$

$$(sg = s', \quad s'h = s'' \implies sgh = s'')$$

This is obviously a category, & has inverses:

$$g: s \rightarrow s' \implies g^{-1}: s' \rightarrow s$$

Note this gives a sensible weakening of the quotient set  $S/G$ . The advantage of  $S//G$  over  $S/G$  is that we put in isomorphisms between elements of  $S$  instead of putting in equations (In the usual quotient set we form equivalence classes, losing all information about how the group action gets us from one element to another in the set.)

Next: define the cardinality of a groupoid  $C$  by:

$$|C| = \sum_{[x] \in \underline{C}} \frac{1}{|\text{Aut}(x)|}$$

where  $\underline{C}$  is the decategorification of  $C$ , i.e. the set of isomorphism classes of objects & for any object  $x \in \underline{C}$ ,  $\text{Aut}(x)$  is the automorphism group of  $x$ , i.e. the group of all isomorphisms  $f: x \rightarrow x$ .

Note:

1)  $|C|$  makes sense if  $C$  is a finite groupoid  
 i.e. one with finitely many objects & morphisms.  
 More generally it makes sense whenever  $C$   
 is tame — i.e. when the sum converges  
 (where we decree  $\frac{1}{|Aut(x)|} = 0$  when  $Aut(x)$  is infinite).

2)  $|C|$  also makes sense for tame categories,  
 i.e. categories for which the sum converges.

Then:

$$|C| = |C_0| \quad \text{underlying groupoid}$$

Theorem: If  $G$  is a finite group acting on the finite  
 set  $S$ ,

$$|S//G| = |S|/|G|$$

Proof:

$$|S//G| = \sum_{[s] \in S/G} \frac{1}{|Stab(s)|}$$

(orbits = isomorphism classes  
 stabilizer = automorphism gp.)

$$= \sum_{[s] \in S/G} \frac{|[s]|}{|G|}$$

$$|[s]| = \frac{|G|}{|Stab[s]|}$$

$$= \frac{1}{|G|} \sum_{[s] \in S/G} |G|$$

$$= \frac{1}{|G|} |S| \quad \blacksquare$$

Now, given a str-type  $F$  and a finite set  $Z_0$ , let

$$F(Z_0) = \sum_n \frac{F_n \times Z_0^n}{n!}$$

where:  $F_n$  is the set of  $F$ -structures on  $n$

$F_n$  has an action of the group  $n!$  because

$F$  is a functor from  $\text{FinSet}_0$  to  $\text{Set}$  so  
 $f: n \xrightarrow{\sim} n$  gives  $F(f): F_n \rightarrow F_n$ .

$Z_0^n$  has an action of  $n!$  via permutation of coordinates.

So:  $F_n \times Z_0^n$  has an action of  $n!$ , so

$$\frac{F_n \times Z_0^n}{n!}$$

is a groupoid.

Finally, given groupoids  $C_n$ , let  $\sum_n C_n$  be their coproduct (disjoint union), again a groupoid



$C_1$



$C_2$



$C_3$

The whole picture is  $\sum_n C_n$ : we just draw them side-by-side.

Clearly :

$$\left| \sum_n C_n \right| = \sum_n |C_n|$$

when  $\sum_n C_n$  is tame, ~~or~~ equivalently when  $\sum_n |C_n|$  converges.

So,  $F(z_0)$  is a groupoid &

$$\begin{aligned} |F(z_0)| &= \left| \sum_n \frac{F_n \times z_0^n}{n!} \right| \\ &= \sum_n \left| \frac{F_n \times z_0^n}{n!} \right| \\ &= \sum_n \frac{|F_n \times z_0^n|}{n!} \leftarrow \text{the number } n! \\ &= \sum_n \frac{|F_n| |z_0|^n}{n!} \\ &= |F| (|z_0|) \end{aligned}$$

as desired