17 Feb 2004

e and the "set" of all finite sets:

groupoid

We've seen how to evaluate a str. type F at a finite set Z_0 and get a groupoid

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n \times Z_0^n}{n!}$$

which has

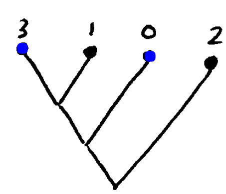
$$|F(Z_0)| = |F|(|Z_0|)$$

when $F(Z_0)$ is <u>tame</u> (cardinality converges). But $F(Z_0)$ makes sense regardless of whether it's tame. What does the groupoid $F(Z_0)$ <u>mean</u>? It's the groupoid of "Z_0-colored, F-structured finite sets." Why?

F_n = set of F-strs on n

$Z_0^n = \{\text{functions } f : n \to Z_0\}$ = set of Z_0-colorings of n

$F_n \times Z_0^n$ = set of pairs consisting of an F-str & a Z_0-coloring of n

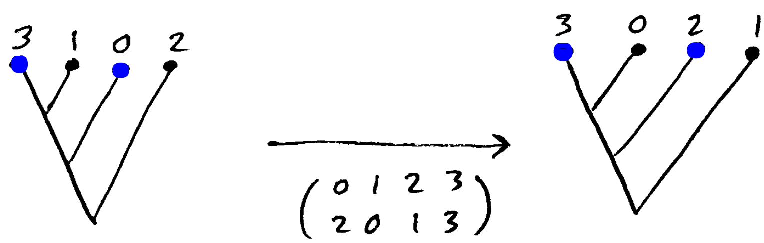
e.g. 

$\in T_4 \times \{\bullet, \bullet\}^4$

$4 = \{0, 1, 2, 3\}$

$$\frac{F_n \times Z_0^{\,n}}{n!} \simeq$$ groupoid of F-structured, Z_0-colored n-elt sets.

- objects are elts of $F_n \times Z_0^{\,n}$
- morphisms $f: x \to y$ is a permutation of n carrying x to y.



$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 3 \end{pmatrix}$$

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n Z_0^{\,n}}{n!} \simeq$$ groupoid of F-structured Z_0-colored finite sets.

Example:
$$e^z = \sum_{n \in \mathbb{N}} \frac{z^n}{n!}$$

is the generating fn of

$$E^Z = \text{ being a finite set}$$

so

$$|E^{Z_0}| = e^{|Z_0|}$$

says the groupoid cardinality of

$$E^{Z_0} \simeq \text{ groupoid of } Z_0\text{-colored finite sets}$$

is $e^{|Z_0|}$.

If $Z_0 = \emptyset$, $E^{Z_0} \simeq$ the groupoid of 0-colored sets = the groupoid w/ $\emptyset$ as its only object & id. morphism of that as only morphism $\simeq \circlearrowright$, and $|E^{Z_0}| = e^{|Z_0|} = e^0 = 1$, and indeed we have $|\circlearrowright| = 1$.

If $Z_0 = 1$, $E^{Z_0} \simeq$ groupoid of 1-colored finite sets $\simeq \mathrm{FinSet}_0$.

So

$$|\mathrm{FinSet}_0| = e^{|1|} = e^1 = e$$

and so, following the French, we should indeed call FinSet_0 "E" ('ensembles').

We usually explain e by saying

$$\frac{df}{dz} = f \quad \& \quad f(0) = 1 \quad \Longrightarrow \quad f(1) = e.$$

Similarly:

$$\frac{D}{DZ} F(Z) \simeq F(Z) \quad \& \quad F(0) = 1 \quad \Longrightarrow \quad F(1) = E.$$

Why? This says: [To put an F-str on S+1 is the same as to put an F-str on S] & [The $\underbrace{\text{groupoid}}$ of F-structured empty sets $\simeq \circlearrowright$]

or just the <u>set</u>, since this particular groupoid has only identity morphisms

$\Longrightarrow$ [The groupoid of F-structured finite sets $\simeq$ The groupoid of finite sets]

So, this says: if F has the property $\frac{DF}{DZ} = F$ and there is only one F structure on $\emptyset$, then there is one ~~str~~ F str. on any finite set. That is $F \simeq$ being a finite set.

CATEGORIFIED HYPERBOLIC TRIGONOMETRY

If $|F|$ is an even function, then F is an even structure, i.e. one that can only be put on even sets. If $|F|$ is odd, we can only put an F-str on odd sets, so F is called an odd structure type.

A very simple even structure is "being an even set." This has generating fn:

$$\sum_{n \in \mathbb{Z}} \frac{z^{2n}}{(2n)!} = \cosh z$$

so define

$$\mathrm{COSH}\, Z = \text{"being an even set"}$$

& similarly

$$\mathrm{SINH}\, Z = \text{"being an odd set"}$$

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Categorified Hyperbolic Trig (cont.)

We saw that "being an even set" deserves the name COSH since its generating function is cosh. Similarly, "being an odd set" should be called SINH. Note that "being a finite set" is E^Z so

$$COSH(Z) + SINH(Z) \cong E^Z$$

just says

"being an even set or an odd set" $\cong$ "being a finite set".

We also have

$$\frac{d}{dz} \cosh z = \sinh z$$

$$\& \quad \frac{d}{dz} \sinh z = \cosh z .$$

Do these come from decategorifying

$$\frac{D}{DZ} COSH\ Z \cong SINH\ Z$$

$$\& \quad \frac{D}{DZ} SINH\ Z \cong COSH\ Z \quad ?$$

Yes! The first one says "putting the structure of being an even set on $S+1$ is the same as putting the structure of being an odd set on S." Similarly for the second. This is just like $\frac{D}{DZ} E^Z = E^Z$, which says "$S+1$ being finite is the same as S being finite."

How about

$$\mathrm{COSH}^2 Z \overset{?}{\cong} \mathrm{SINH}^2 Z + 1$$

To put a COSH^2-str. on S, chop S into 2 even sets.

e.g. $S = 2 = \{0,1\}$

A	B
0 1	

A	B
	0 1

To put a SINH^2-str. on S, either put the str 1 ("being the empty set") on S or chop S into two odd sets

e.g. $S = 2 = \{0,1\}$

A	B
0	1

A	B
1	0

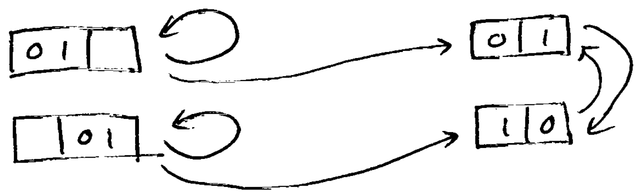
Alas, this is false – there's no natural isomorphism between the functors

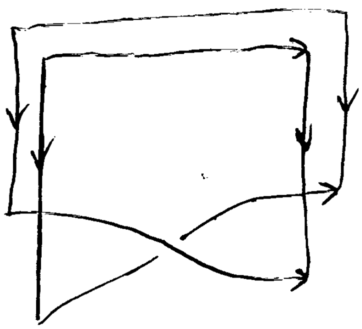
$$\begin{tikzcd} \mathrm{FinSet}_0 \arrow[r, bend left, "\mathrm{COSH}^2"] \arrow[r, bend right, "\mathrm{SINH}^2 + 1"'] \arrow[r, phantom, "\alpha \Downarrow \wr"] & \mathrm{Set} \end{tikzcd} \qquad (\not\exists)$$

If there were such an α,

$$\begin{tikzcd} (\mathrm{COSH}^2)_2 \arrow[r, "(\mathrm{COSH}^2)_\sigma"] \arrow[d, "\alpha_2"', "\wr"] & (\mathrm{COSH}^2)_2 \arrow[d, "\alpha_2"', "\wr"] \\ (\mathrm{SINH}^2 + 1)_2 \arrow[r, "(\mathrm{SINH}^2 + 1)_\sigma"'] & (\mathrm{SINH}^2 + 1)_2 \end{tikzcd} \qquad \sigma \in 2!$$

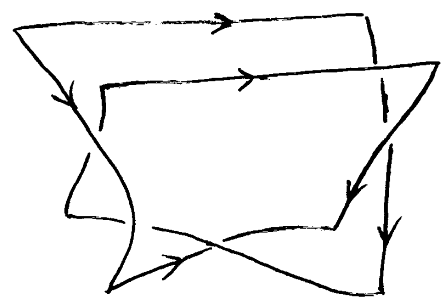
would have to commute for all bijections $\sigma : 2 \to 2$. But it doesn't when $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the nonidentity permutation in $2!$. Reason: switching "0" & "1" switches the two SINH^2+1-strs on 2 but not COSH^2.





doesn't commute.

nor does:



But we do have natural isos:

$$\frac{D}{DZ}\, \mathrm{COSH}^2 Z \cong \frac{D}{DZ}\, \mathrm{SINH}^2 Z + 1$$

and

$$\mathrm{COSH}^2\, 0 \cong \mathrm{SINH}^2\, 0 + 1$$

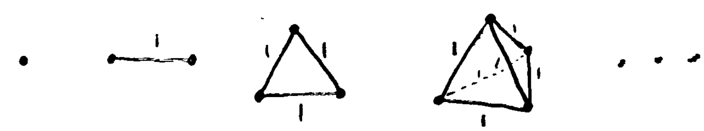
<u>Moral</u>: We can't generally "integrate" str types to show:

$$F' \cong G' \;\&\; F(0) = G(0) \implies F \cong G$$

HALF-COLORED SETS, DOUBLE FACTORIALS, HYPERCUBES, & $\sqrt{e}$

Let "Simplices" be the groupoid whose objects are fin.-dim. regular simplices:

the -1 simplex is $\emptyset$



with sides of length 1 & whose morphisms are metric preserving bijections. Note:

$$\text{Simplices} \simeq \text{FinSet}_0$$

since automorphisms of the n-dim simplex are permutations of its n+1 vertices. So

$$|\text{Simplices}| = e.$$

Now let "Cubes" be the groupoid whose objects are fin-dim unit cubes:

I^{-1} $\quad I^0$ $\quad I^1$ $\quad I^2$ $\quad I^3$

where $I = [-\frac{1}{2}, \frac{1}{2}]$ and morphisms are metric-preserving bijections. What's $|\text{Cubes}|$?

$$|\text{Cubes}| = \sum_{[x] \in \underline{\text{Cubes}}} \frac{1}{|\text{Aut}(x)|}$$

So what's $|\text{Aut}(I^n)|$

$$\begin{aligned}
&\text{Aut}(\bullet) \cong 1 && \text{so} \quad |\text{Aut}(\bullet)| = 1 \\
&\text{Aut}(\bullet\!-\!\bullet) \cong \mathbb{Z}/2 && |\text{Aut}(\bullet\!-\!\bullet)| = 2 = 1!\,2^1 \\
&\text{Aut}(\square) \cong D_4 && |\text{Aut}(\square)| = 8 = 2!\,2^2 \\
&\text{Aut}(⧉) \cong S_4 \times \mathbb{Z}_2 && |\text{Aut}(⧉)| = 48 = 3!\,2^3 \\
&\phantom{\text{Aut}(⧉)} \cong S_3 \ltimes \mathbb{Z}_2^3 &&
\end{aligned}$$

In general

reflections along axes — permutation of axes

$$1 \longrightarrow \mathbb{Z}_2^n \hookrightarrow \text{Aut}(I^n) \twoheadrightarrow S_n \longrightarrow 1$$

we have a short exact sequence, so:

$$\begin{aligned}
|\text{Aut}(I^n)| &= |\mathbb{Z}_2^n|\,|S_n| \\
&= 2^n n! \\
&= n!!
\end{aligned}$$

So

cubes : double factorials :: simplices : factorials

and

$$\begin{aligned}
|\text{Cubes}| &= \sum_{n \in \mathbb{N}} \frac{1}{(2n)!!} \qquad \left(\text{like } |\text{Simplices}| = \sum_{n \in \mathbb{N}} \frac{1}{n!}\right) \\
&= \sum_{n \in \mathbb{N}} \frac{1}{2^n n!} \\
&= \sqrt{e}
\end{aligned}$$

So:

$$|\text{Simplices}| = |\text{Cubes}|^2$$

In what sense, if any, is it true that

$$\text{Cubes} = \sqrt{\text{Simplices}} \quad ?$$

We'll see that just as

$$\text{Simplices} \simeq \text{FinSet}_0 \simeq E^1,$$

$$\text{Cubes} \simeq E^{\frac{1}{2}},$$

the groupoid of "$\frac{1}{2}$-colored sets."