Review:

We defined the cardinality $|C|$ of a groupoid C by:

$$\sum_{[x] \in \underline{C}} \frac{1}{|\mathrm{Aut}(x)|}$$

E.g. if C is groupoid with one object, x, then C is really just a group $G = \mathrm{Aut}(x)$ &

$$|C| = \frac{1}{|G|}$$

usual cardinality of G.

The group $G = \mathbb{Z}_n$ gives a groupoid C with $|C| = \frac{1}{n}$ this way.

If F is a structure type & Z_0 is a set, we defined $F(Z_0)$ to be the groupoid of "F-structured, Z_0-colored sets," also given by:

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n \times Z_0^n}{n!}.$$

the weak quotient by the permutation group $n!$ - a groupoid

Here $F_n \times Z_0^n$ is the cartesian product of F_n and n copies of Z_0, i.e. the set of all "F-structures and Z_0-colorings" on n; $\frac{F_n \times Z_0^n}{n!}$ is the groupoid (the weak quotient) we get from the orbits of a particular $n!$-action on $F_n \times Z_0^n$ (see p. 58). So $F(Z_0)$ is the sum (disjoint union) of all of these groupoids one for each $n \in \mathbb{N}$

We also noted:

$$|F(Z_0)| = |F|(|Z_0|)$$

Example: If our structure type is "being a finite set" we call it E^Z since

$$|E^Z| = e^Z.$$

Then if Z_0 is some set, E^{Z_0} is the groupoid of Z_0-colored finite sets. Note:

$$|E^{Z_0}| = e^{|Z_0|}$$

so e.g. the groupoid of 17-colored finite sets has cardinality e^{17}.

But, we saw the groupoid Cubes has cardinality $e^{1/2}$. This suggests that we generalize this game to define $F(Z_0)$ when Z_0 is a groupoid & take Z_0 to be the groupoid corr. to $\mathbb{Z}/2$.

So: given a structure type F & groupoid Z_0, we will try to define a groupoid

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n \times Z_0^n}{n!}$$

Note: if C & D are groupoids we have a groupoid $C \times D$ (whose objects are ordered pairs in the obvious way, and same for morphisms) and $|C \times D| = |C||D|$.

So Z_0^n is the groupoid $\underbrace{Z_0 \times \cdots \times Z_0}_{n}$, and $|Z_0^n| = |Z_0|^n$.

Also, given a set S, we can think of it as a discrete groupoid, i.e. one with only identity morphisms and

$$\underset{\text{groupoid cardinality}}{|S|} = \underset{\text{set cardinality}}{|S|}$$

So: $F_n \times Z_0^n$ makes sense as a groupoid &

$$|F_n \times Z_0^n| = |F_n||Z_0|^n.$$

Now: what is the groupoid $\dfrac{F_n \times Z_0^n}{n!}$? If we can define this so that

$$\left| \frac{F \times Z_0^n}{n!} \right| = \frac{|F_n \times Z_0^n|}{n!} \leftarrow \text{the number}$$

then we'll get:

$$\begin{aligned} |F(Z_0)| &= \left| \sum_{n \in \mathbb{N}} \frac{F_n \times Z_0^n}{n!} \right| \\ &\overset{?}{=} \sum_{n \in \mathbb{N}} \frac{|F_n \times Z_0^n|}{n!} \\ &= \sum_{n \in \mathbb{N}} \frac{|F_n||Z_0|^n}{n!} \\ &= |F|(|Z_0|) \end{aligned}$$

If we want this to work, we need to define what it means for a group G to act on a groupoid (or more generally, on a category) C & define the weak quotient $C/\!/G$, which is a groupoid (or a category) with

$$|C/\!/G| = |C|/|G|$$

<u>Def</u>: A (<u>strict</u>) <u>action</u> A of a group G on a category C is: for each $g \in G$, a functor

$$A(g) : C \longrightarrow C$$

s.t.

$$A(g)A(h) = A(gh)$$
$$A(1) = 1_C$$

(Note: this is "strict" because it involves equations between functors, rather than natural isomorphisms. There are also weak actions, but we don't need them now since $n!$ acts strictly on $F_n \times Z_0^n$.)

Now for the weak quotient $C/\!/G$ - here we really need weakness to get $|C/\!/G| = |C|/|G|$.

Recall the usual quotient of a set by a group. Suppose A is an action of a group G on a set S. What is S/G, <u>really</u>? We can characterize it by a universal property: a quotient of S by G is a set X equipped with a "quotient map" $j: S \to X$, such that:

$$\begin{tikzcd} S \arrow[rr, "A(g)"] \arrow[dr, "j"'] & & S \arrow[dl, "j"] \\ & X & \end{tikzcd}$$

commutes and $j: S \to X$ is initial among all such. I.e. if $j': S \to X'$ makes

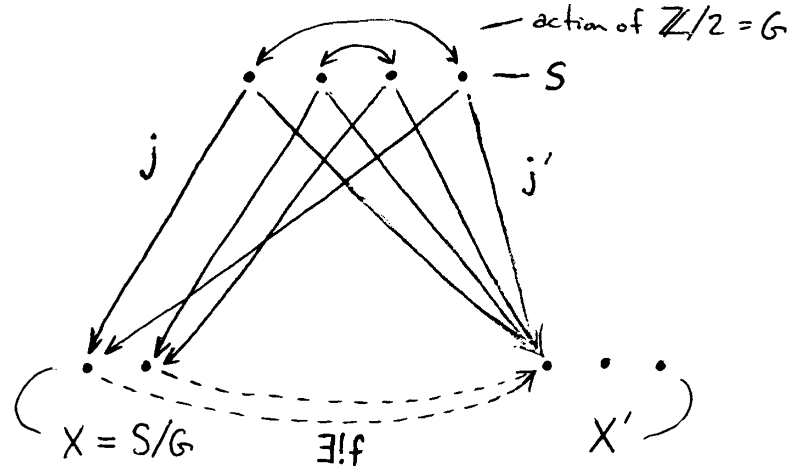
$$\begin{tikzcd} S \arrow[rr, "A(g)"] \arrow[dr, "j'"'] & & S \arrow[dl, "j'"] \\ & X' & \end{tikzcd}$$

commute, then $\exists!\ f: X \to X'$ s.t.

$$\begin{tikzcd} & S \arrow[dl, "j"'] \arrow[dr, "j'"] & \\ X \arrow[rr, "f"'] & & X' \end{tikzcd}$$

commutes.

Example: $S = 4$, $G = \mathbb{Z}/2$ acting via reflection:



Now, let A be an action of a group G on a category C. We want to define the weak quotient by a similar universal property ... but weakened?

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27 Feb 2004

Weak Quotients

Suppose we have a (strict) action A of a group G on a category C. We want to define the "weak quotient" $C//G$ – a category, but a groupoid when C is a groupoid, and reducing to our old "$S//G$" when C is a discrete groupoid (secretly just a set S). We'll define it by a "weak universal property" – replacing equations in last lecture's universal property by natural isomorphisms – but imposing "coherence laws" asserting that potentially different isos are equal. This happens all the time in categorification.

Def – A <u>weak quotient</u> of a category C by some group G having some action A on it consists of a category $C /\!/ G$ together with a <u>quotient functor</u>

$$j : C \to X$$

such that

$$\begin{tikzcd} C \arrow[rr, "A(g)"] \arrow[dr, "j"'] & \arrow[d, phantom, "\overset{\tau(g)}{\underset{\sim}{\swarrow}}"] & C \arrow[dl, "j"] \\ & X & \end{tikzcd}$$

commutes up to natural isomorphism $\tau(g)$. That is, for any object $c \in C$ we get an isomorphism

$$j\, A(g)(c) \xrightarrow[\sim]{\tau(g)_c} j(c)$$

such that $\forall f : c \to c'$,

$$\begin{tikzcd} j\,A(g)(c) \arrow[r, "\tau(g)_c", "\sim"'] \arrow[d, "jA(g)(f)"'] & j(c) \arrow[d, "j(f)"] \\ jA(g)(c') \arrow[r] & j(c') \end{tikzcd}$$

commutes.

We demand the coherence law:

$\forall g, h \in G$

$$
\begin{tikzcd}
C \arrow[rr, "A(gh)"] \arrow[dr, "A(g)"'] \arrow[ddr, bend right, "j"'] \arrow[rr, phantom, shift right=4ex, "//"] & & C \arrow[ddl, bend left, "j"] \\
& C \arrow[ur, "A(h)"'] \arrow[d, "j"] \arrow[ur, phantom, shift left=3ex, "\Swarrow \tau(gh)"] \arrow[dl, phantom, "\Swarrow \tau(g)"] \arrow[dr, phantom, "\Swarrow \tau(h)"] & \\
& X &
\end{tikzcd}
$$

This tetrahedron commutes, i.e.

$$
\begin{tikzcd}
C \arrow[r, "A(g)"] \arrow[dr, "j"'] \arrow[r, phantom, shift right=4ex, "\Swarrow \tau(g)"] & C \arrow[r, "A(h)"] \arrow[d, "j"] \arrow[r, phantom, shift right=4ex, "\Swarrow \tau(h)"] & C \arrow[dl, "j"] \\
& X &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[rr, "A(gh)"] \arrow[dr, "j"'] \arrow[rr, phantom, shift right=4ex, "\Swarrow \tau(gh)"] & & C \arrow[dl, "j"] \\
& X &
\end{tikzcd}
$$

(front faces) (back face)

i.e. the right-hand side gives for any $c \in C$ a morphism

$$jA(gh)(c) \xrightarrow{\tau(gh)_c} j(c)$$

while the left-hand side gives another

$$jA(h)A(g)(c) \xrightarrow{\tau(h)_{A(g)c}} jA(g)(c) \xrightarrow{\tau(g)_c} j(c),$$

and the coherence law says these are equal.

Furthermore we demand that X satisfy a universal property, namely that it be "weakly initial." I.e., given any other

$$C \xrightarrow{j'} X'$$

with

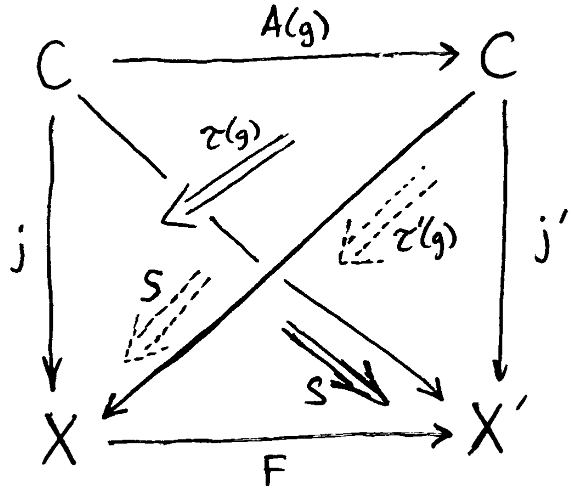
$$\begin{tikzcd} C \arrow[rr, "A(g)"] \arrow[dr, "j'"'] & \arrow[d, phantom, "\overset{\tau'(g)}{\Swarrow \sim}"] & C \arrow[dl, "j'"] \\ & X' & \end{tikzcd}$$

making an analogous tetrahedron commute (i.e. given any "competitor") there exists $F : X \to X'$ such that

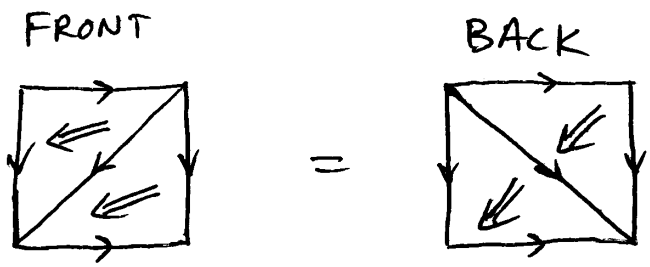
$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] \arrow[d, phantom, "\overset{S}{\Swarrow \sim}"] & \\ X \arrow[rr, "F"'] & \, & X' \end{tikzcd}$$

commutes up to natural isomorphism S

... and :



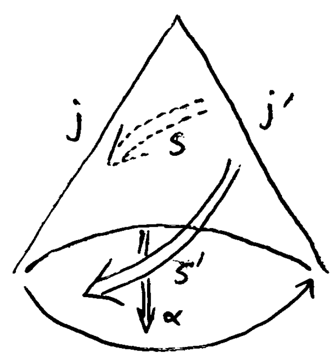
commutes! I.e., this coherence law holds:



Moreover, F is "weakly unique," i.e. given any other $F' : X \to X'$ with this property (i.e. a "competitor") there exists a unique natural isomorphism

$$\begin{tikzcd} X \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & X' \arrow[Rightarrow, from=U, to=D, "\wr", "\alpha"'] \end{tikzcd}$$

such that this cone commutes:



— END OF DEFINITION —

Moral: categorical universal properties are like a game where you say "for any competitor, I can find a map F, s.t. for any competitor F', I can find an $\alpha : F \Rightarrow F'$, s.t. ..."

Being universal is like having a winning strategy in a 2-person game – Michael Makkai.

(i.e. "I can make a move such that, for any move you make, I can make a move such that, ~~such that~~ for any move ... I win.")