The weak quotient \( C//G \) of a category by a group acting on it.

While defining weak quotients was a bit scary, calculating them is easier, & understanding them is (eventually) easier still.

Suppose \( A \) is a (strict) action of a group \( G \) on a category \( C \). What's \( C//G \) like? Recall that \( C//G \) comes with the quotient map (functor)

\[
j : C \to C//G
\]

and natural isos

\[
\begin{array}{ccc}
C & \xrightarrow{A(g)} & C \\
\downarrow{j} & & \downarrow{j} \\
C//G & & C//G
\end{array}
\]

such that this tetrahedron commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{A(gh)} & C \\
\downarrow{j} & & \downarrow{j} \\
\downarrow{j} & & \downarrow{j} \\
\downarrow{j} & & \downarrow{j} \\
C//G & & C//G
\end{array}
\]

... and it should be "weakly universal" among such gizmos.
So to get $C//G$, $j$, $\tau(g)$ we should create it "freely." Therefore, just take $C//G$ to be the category with the same objects and all the same morphisms too, but with extra ones thrown in, as required by our definition, like this:

$$j(c) \xrightarrow{\tau(g)c} j(A(g)c)$$

(which we demand are isos, so they come with inverses).
To make $\tau(g)$ natural, we impose relations saying these squares commute:

$$\begin{array}{c}
\begin{array}{ccc}
j(c) & \xrightarrow{\tau(g)c} & j(A(g)c) \\
\downarrow j(f) & & \downarrow j(A(g)f) \\
j(c') & \xrightarrow{} & j(A(g)c')
\end{array}
\end{array}$$

1) We also impose relations to make the tetrahedron commute:

$$\begin{array}{c}
\begin{array}{ccc}
j(c) & \xrightarrow{\tau(gh)c} & j(A(gh)c) \\
\downarrow & & \downarrow \\
\tau(g)c & \xrightarrow{} & \tau(h)A(g)c
\end{array}
\end{array}$$

commutes.

We define $j : C \to C//G$ to be the obvious functor. Also define $\tau(g)$'s to be the obvious natural isomorphisms, mapping $c \in C$ to $\tau(g)_c$. 
Theorem: This construction gives a weak quotient of $C$ by $G$.

Proof: Too boring to prove! 

Examples:

A) Suppose $C$ is the "discrete category" on the set $S$ - elts of $S$ are its objects & there are only identity morphisms. Then $C/G$ reduces to $S/G$ as defined earlier, since (1) becomes trivial & (2) is the old relation written in a new way.

B) Suppose $C$ is a groupoid with one object, so the morphisms form a group $H$. Then an action of $G$ on $C$ is just an action of $G$ on $H$, and $C/G$ is secretly just $G \times H$! That is, $C/G$ is a one-object groupoid - i.e. a group - gen by elts. of $H$ together with elements $\tau(g)$ satisfying relations:

1) $\tau(g) h = (A(g) h) \tau(g)$
2) $\tau(g) \tau(g') = \tau(gg')$

This is just the definition of the semi-direct product, $G \ltimes H$! (Note: 1) $\iff A(g) h = \tau(g) h \tau(g)^{-1}$
Moral: semidirect products are weak quotients.

**Thm.** \[ |G//G| = |C/1G| \]

**Proof:** A tiresome calculation, but for examples of type A we've already checked it:
\[ |S//G| = |S|/|G| \]
and for examples of type B it's also easy: if C is a one-object groupoid with morphisms forming the group H, then
\[ |C//G| = \frac{1}{|G\times H|} \quad \text{since } C//G \text{ has one object & } G\times H \text{ as automorphisms} \]
\[ = \frac{1}{|G||H|} \quad \text{since } |C| = \frac{1}{|H|} \]
\[ = \frac{|C|}{|G|} \]

**Example:** (Back to our motivating example)

Let F be the structure type "being a finite set" and \(Z_0\) be the groupoid with one object and \(Z/2\) as the automorphisms of this object.

\[ F(Z_0) = \text{"the group of } Z_0\text{-colored, } F\text{-structured finite sets"} \]
\[ \text{"} \frac{1}{2} \text{ colored" in our case} \]
Or, since $F$ is vacuous:

$$F(Z_0) \cong \text{"the groupoid of } \frac{1}{2} \text{-colored finite sets."}$$

Does this really make sense? $|Z_0| = \frac{1}{2}$, but what’s a $Z_0$-colored finite set, much less a $\frac{1}{2}$-colored finite set? More honestly:

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n \times Z_0^n}{n!}$$

$$= \sum_{n \in \mathbb{N}} \frac{Z_0^n}{n!} \quad \text{since } F_n \cong 1.$$

$Z_0^n$ is a groupoid with one object, i.e. secretly a group, namely: $(\mathbb{Z}/2)$. $n!$ acts on this by permutations, so $Z_0^n/n!$ is a one-object groupoid that’s secretly $S_n \times (\mathbb{Z}/2)^n$, the group of symmetries of an $n$-cube! So:

$$F(Z) \cong \text{Cubes}$$

the groupoid of $n$-cubes.

Note $F(Z) = E^Z$ so

$$|\text{Cubes}| = |E^Z|$$

$$= |E|^{|Z|}$$

$$= e^{\frac{1}{2}} = \sqrt{e} \quad \text{as we saw before.}$$
If this is a regular 5-element set, its automorphism group is $5!$. If it's a "$\frac{1}{2}$-colored 5-elt set," its automorphism group is $5! \times (\mathbb{Z}/2)^5$ (with $5! \times 2^5$ elements) so the 5 dots can be "flipped over" giving extra internal symmetries.

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Given a structure type $F$ & a groupoid $Z_0$, we define the groupoid

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n \times Z_0^n}{n!}.$$ 

We know

$$|F(Z_0)| = |F|(1 |Z_0|)$$

but can we understand $F(Z_0)$ intuitively as:

"the groupoid of $F$-structured, $Z_0$-colored finite sets"

as we did in the case when $Z_0$ was a mere set?
Or better:

"the groupoid of $F$-structured finite sets labelled by objects of $Z_0"$.

Yes! Let's see how this works. Recall the weak quotient

$$\frac{F_n \times Z_0^n}{n!}$$

has the same objects as $F_n \times Z_0^n$, so its objects of this is:

an ordered pair consisting of

- an $F$-structure on the n-elt set
- an $n$-tuple of objects in $Z_0$

i.e.

an $F$-structure on $n$ & a way of labelling its elements by objects of $Z_0$.

Example:

$F = \text{being a totally ordered set}$

$$Z_0 = \begin{array}{c}
1 \xrightarrow{f} G \\
G \xrightarrow{f^*} 1
\end{array}$$

$g \in B^{18}, \quad g^2 = 18$

$n = 0,1,2,33 = 4$

An object of $\frac{F_4 \times Z_0^4}{4!} : \quad R \quad B \quad R \quad G$
The morphisms of \( \frac{F_n \times Z_0}{n!} \) are generated by morphisms in \( F_n \times Z_0 \) & morphisms coming from elts of the permutation group \( n! \), & satisfying certain relations we called (1) & (2) last time. In our example, a typical morphism in \( F_n \times Z_0 \) is

\[
\begin{array}{cccc}
3 & 1 & 0 & 2 \\
R & B & R & G \\
f & g & 1 & f' \\
G_0 & B & R & R \\
3 & 1 & 0 & 2
\end{array}
\]

while a typical morphism coming from \( 4! \) is:

\[
\begin{array}{cccc}
6 & B & R & R \\
3 & 1 & O & 2 \\
\end{array}
\]

\[
\begin{array}{cccc}
G_0 & B & R & G_0 \\
3 & 1 & 0 & 2 \\
\end{array}
\]

--- a permutation in \( 4! \)---

The relations (1) & (2) are obvious in this picture, e.g.,

\[
\begin{array}{cccc}
B & R & B & R \\
6 & 8 & 6 & 8 \\
g & f & g & f \\
\end{array}
\]

is an example of relation (2).
So this is what we mean when we say

\[ \frac{F_n \times Z_0^n}{n!} = \text{"the groupoid of } F\text{-structured, } \ Z_0\text{-colored } n\text{-elt. sets"} \]

& similarly:

\[ F(\mathbb{Z}_0) = \sum_{n \in \mathbb{N}} \frac{F_n \times \mathbb{Z}_0^n}{n!} = \text{"the groupoid of } F\text{-structured, } \mathbb{Z}_0\text{-colored finite sets"} \]

More examples:

\[ E^\mathbb{Z} = \text{"being a finite set"} \]

so if \( \mathbb{Z}_0 \) is a groupoid,

\[ E^\mathbb{Z}_0 = \text{"the groupoid of } \mathbb{Z}_0\text{-colored finite sets"} = \text{"the groupoid of finite sets labelled by } \mathbb{Z}_0\text{-objects"} \]

Similarly:

\[ \text{new notation } \frac{Z^k}{k!} = \text{"being a } k\text{-elt. set"} \rightarrow \frac{Z_0^k}{k!} = \text{"the groupoid of } \mathbb{Z}_0\text{-colored } k\text{-elt. sets"} \text{ when } \mathbb{Z}_0 \text{ is a groupoid.} \]

\[ E^\mathbb{Z}_0^k/k! = \text{"the groupoid of finite sets labelled by } \mathbb{Z}_0\text{-colored } k\text{-elt. sets"} \]
If $Z_0 = 1$ (the set $I$ with only the identity morphism)

$$E^{1/k!} = \text{the groupoid of finite sets}$$

labelled by $k$-element sets.

A morphism in here is like this

A bundle of $3$-element sets over a $4$-element set

i.e. a permutation with strands labelled by morphisms in the groupoid $1/3!$ (w. one object and permutations in $3!$ as morphisms)

What's the cardinality of $E^{1/3!}$?

$$|E^{1/3!}| = e^{1/3!} = e^{1/13!} = e^{1/6}$$
Now, how about:

\[ E^{(E^{Z_0})} = \text{"the groupoid of finite sets labelled by } \mathbb{Z}_0\text{-colored finite sets."} \]

or if \( Z_0 = 1 \):

\[ E^{E^1} = E^E = \text{"the groupoid of finite sets labelled by finite sets."} \]

A morphism

objects here are "finite set bundles over finite sets." The cardinality of this groupoid is

\[ |E^E| = e^e \approx 15,154 \]