2 March 2004

The weak quotient $C/\!\!/G$ of a category by a group acting on it.

While <u>defining</u> weak quotients was a bit scary,
<u>calculating</u> them is easier, &
<u>understanding</u> them is (eventually) easier still.

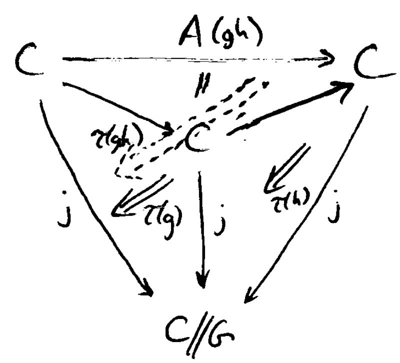
Suppose A is a (strict) action of a group G on a category C. What's $C/\!\!/G$ like? Recall that $C/\!\!/G$ comes with the quotient map (functor)

$$j : C \longrightarrow C/\!\!/G$$

and natural isos

$$\begin{tikzcd} C \arrow[rr, "A(g)"] \arrow[dr, "j"'] & \arrow[d, phantom, "\overset{\tau(g)}{\Leftarrow}\sim"] & C \arrow[dl, "j"] \\ & C/\!\!/G & \end{tikzcd}$$

such that this tetrahedron commutes:



... and it should be "weakly universal" among such gizmos.

So to get $C/\!\!/G$, j, $\tau(g)$ we should create it "freely". So just take $C/\!\!/G$ to be the category with the same objects and all the same morphisms too, but with extra ones thrown in, as required by our definition, like this:

$$j(c) \xrightarrow[\sim]{\tau(g)_c} j(A(g)c)$$

(which we demand are isos, so they come with inverses). To make $\tau(g)$ natural, we impose relations saying these squares commute:

1)
$$\begin{tikzcd}
j(c) \arrow[r, "\tau(g)_c"] \arrow[d, "j(f)"'] & j(A(g)c) \arrow[d, "j(A(g)f)"] \\
j(c') \arrow[r] & j(A(g)c')
\end{tikzcd}$$

& we also impose relations to make the tetrahedron commute:

2)
$$\begin{tikzcd}
j(c) \arrow[rr, "\tau(gh)_c"] \arrow[dr, "\tau(g)_c"'] & & j(A(gh)(c)) \\
& j\,A(g)(c) \arrow[ur, "\tau(h)_{A(g)(c)}"'] &
\end{tikzcd}$$
commutes.

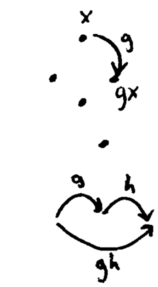
We define $j : C \longrightarrow C/\!\!/G$ to be the obvious functor. Also define $\tau(g)$'s to be the obvious natural isomorphisms, mapping $c \in C$ to $\tau(g)_c$.

Theorem: This construction gives a weak quotient of C by G

Proof: Too boring to prove! ∎

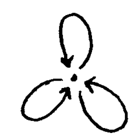
Examples:

A) Suppose C is the "discrete category" on the set S — elts of S are its objects & there are only identity morphisms. Then $C/\!/G$ reduces to $S/\!/G$ as defined earlier, since (1) becomes trivial & (2) is the old relation, written in a new way.



B) Suppose C is a groupoid with one objects, so the morphisms form a group H. Then an action of G on C is just an action of G on H, and $C/\!/G$ is secretly just $G \ltimes H$! That is, $C/\!/G$ is a one-object groupoid – i.e. a group – gen by elts. of H together with elements $\tau(g)$ satisfying relations:

Example



$H = \mathbb{Z}/3$

G acting on C

1) $\tau(g)\,h = (A(g)h)\,\tau(g)$
2) $\tau(g)\,\tau(g') = \tau(gg')$

This is just the definition of the semidirect product, $G \ltimes H$! (Note: 1) $\Longleftrightarrow A(g)(h) = \tau(g)\,h\,\tau(g)^{-1}$

Moral: semidirect products are weak quotients.

Thm: $|G/\!/G| = |C|/|G|$

Proof: A tiresome calculation, but for examples of type A we've already checked it:

$$|S/\!/G| = |S|/|G|$$

and for examples of type B it's also easy: if C is a one-object groupoid with morphisms forming the group H, then

$$|C/\!/G| = \frac{1}{|G \ltimes H|}$$

since $C/\!/G$ has one object & $G \ltimes H$ as automorphisms

$$= \frac{1}{|G|\,|H|}$$

since $|C| = \frac{1}{|H|}$

$$= \frac{|C|}{|G|} \quad \blacksquare$$

Example: (Back to our <u>motivating</u> example)

Let F be the structure type "being a finite set" and Z_0 be the groupoid with one object and $\mathbb{Z}/2$ as the automorphisms of this object.

$$F(Z_0) \stackrel{?}{=} \text{"the group of } \underbrace{Z_0\text{-colored}}_{\text{"}\frac{1}{2}\text{ colored" in our case}}\text{, F-structured finite sets"}$$

Or, since F is vacuous:

$$F(Z_0) \stackrel{?}{=} \text{"the groupoid of } \tfrac{1}{2}\text{-colored finite sets."}$$

Does this really make sense? $|Z_0| = \frac{1}{2}$, but what's a Z_0-colored finite set, much less a $\frac{1}{2}$-colored finite set? More honestly:

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n \times Z_0^n}{n!}$$

$$= \sum_{n \in \mathbb{N}} \frac{Z_0^n}{n!} \qquad \text{since } F_n \cong 1.$$

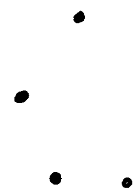
Z_0^n is a groupoid with one object, i.e. secretly a group, namely: $(\mathbb{Z}/2)^n$. $n!$ acts on this by permutations, so $Z_0^n /\!/ n!$ is a one-object groupoid that's secretly $S_n \ltimes (\mathbb{Z}/2)^n$, the group of symmetries of an n-cube! So:

$$F(Z_0) \simeq \text{Cubes}$$

the groupoid of n-cubes.

Note $F(Z) = E^Z$ so

$$|\text{Cubes}| = |E^{Z_0}|$$
$$= |E|^{|Z_0|}$$
$$= e^{\frac{1}{2}} = \sqrt{e} \quad \text{as we saw before.}$$



If this is a regular 5-element set, its automorphism group is $5!$. If it's a "$\frac{1}{2}$-colored 5-elt set," its automorphism group is $5! \ltimes (\mathbb{Z}/2)^5$ (with $5! \times 2^5$ elements) so the 5 dots can be "flipped over" giving extra internal symmetries.

---

4 March 2004

Given a structure type F & a groupoid Z_0, we define the groupoid

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n \times Z_0^n}{n!} .$$

We know

$$|F(Z_0)| = |F|(|Z_0|)$$

but can we understand $F(Z_0)$ intuitively as:

"the groupoid of F-structured, Z_0-colored finite sets"

as we did in the case when Z_0 was a mere set?

Or better:

"the groupoid of F-structured finite sets labelled by objects of Z_0."

Yes! Let's see how this works. Recall the weak quotient

$$\frac{F_n \times Z_0^{\,n}}{n!}$$

has the same objects as $F_n \times Z_0^{\,n}$, so ~~its~~ an object~~s~~ of this is:

an ordered pair consisting of
an F-structure on the n-elt set
& an n-tuple of objects in Z_0

i.e.

an F-structure on n & a way of labelling its elements by objects of Z_0.

<u>Example</u>:

F = being a totally ordered set

$$Z_0 = \begin{tikzcd} R \arrow[loop left, "1_R"] \arrow[r, bend left, "f"] & G \arrow[l, bend left, "f^{-1}"] \arrow[loop right, "1_G"] \\ B \arrow[loop left, "g"] \arrow[loop right, "1_B"] & \end{tikzcd} \qquad g^2 = 1_B$$

$$n = \{0,1,2,3\} = 4$$

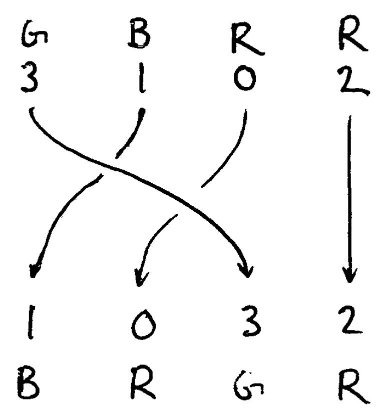
An object of $\dfrac{F_4 \times Z_0^{\,4}}{4!}$:

3	1	0	2
R	B	R	G

The morphisms of $\frac{F_n \times Z_0^n}{n!}$ are generated by morphisms in $F_n \times Z_0^n$ & morphisms coming from elts of the permutation group $n!$, & satisfying certain relations we called (1) & (2) last time. In our example, a typical morphism in $F_n \times Z_0^n$ is

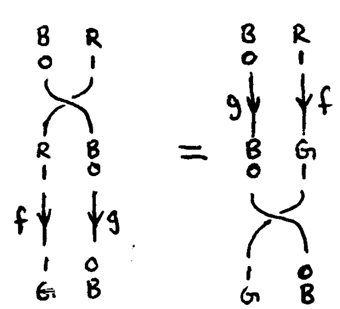
$$\begin{tikzcd}[row sep=small]
3 & 1 & 0 & 2 \\
R \arrow[d, "f"'] & B \arrow[d, "g"'] & R \arrow[d, "1_R"'] & G \arrow[d, "f^{-1}"'] \\
G & B & R & R \\
3 & 1 & 0 & 2
\end{tikzcd}$$

while a typical morphism coming from $4!$ is:



— a permutation in $4!$

The relations (1) & (2) are obvious in this picture, e.g.:



is an example of relation (2).

So this is what we mean when we say

$$\frac{F_n \times Z_0^n}{n!} = \text{“the groupoid of } F\text{-structured, } Z_0\text{-colored } n\text{-elt. sets”}$$

& similarly:

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n \times Z_0^n}{n!} = \text{“the groupoid of } F\text{-structured } Z_0\text{-colored finite sets”}$$

More examples:

$$E^Z = \text{“being a finite set”}$$

so if Z_0 is a groupoid,

$$E^{Z_0} = \text{“the groupoid of } Z_0\text{-colored finite sets”}$$
$$= \text{“the groupoid of finite sets labelled by } Z_0\text{-objects”}$$

Similarly:

new notation $\longrightarrow$ $$\frac{Z^k}{k!} = \text{“being a } k\text{-elt. set”}$$

so

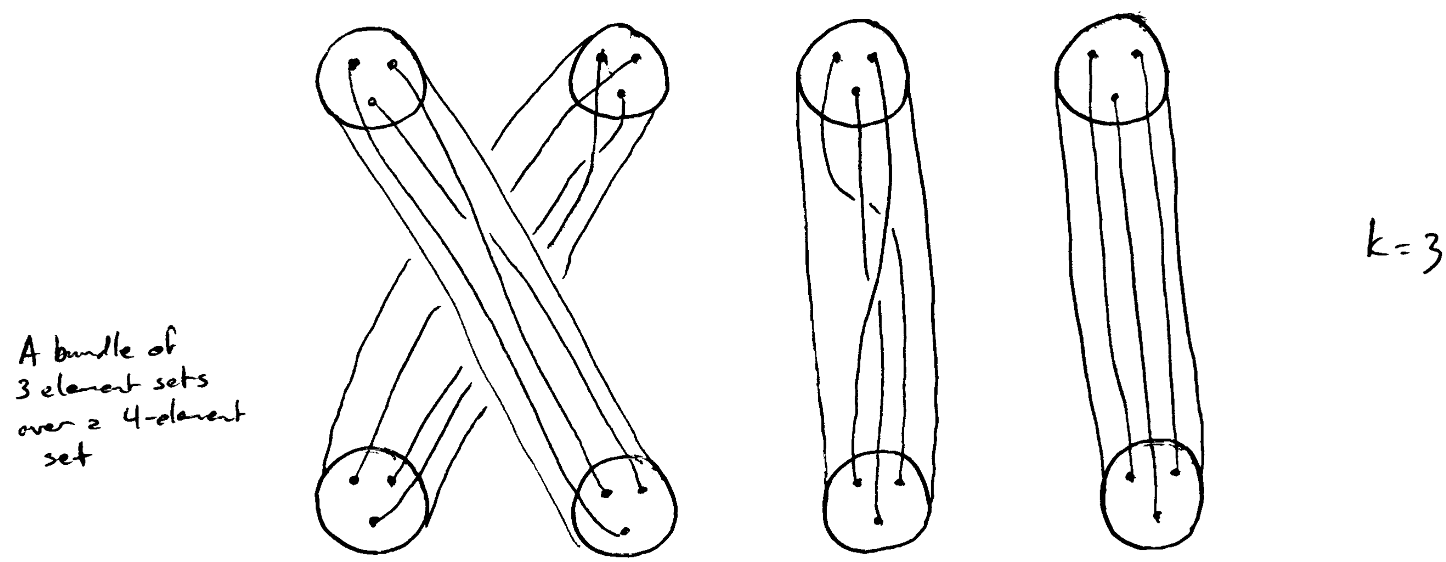
$$\frac{Z_0^k}{k!} = \text{“the groupoid of } Z_0\text{-colored } k\text{-elt sets”}$$ when Z_0 is a groupoid.

$$E^{Z_0^k /\!/ k!} = \text{“the groupoid of finite sets labelled by } Z_0\text{-colored } k\text{-elt. sets”}$$

If $Z_0 = 1$ (the set 1 with only the identity morphism)

$E^{1/\!/k!}$ = the groupoid of finite sets labelled by k-elt. sets.

A morphism in here is like this



i.e. a permutation with strands labelled by morphisms in the groupoid $1/\!/3!$ (w. one object and permutations in $3!$ as morphisms)

What's the cardinality of $E^{1/\!/3!}$?

$$|E^{1/\!/3!}| = e^{|1/\!/3!|} = e^{1/|3!|} = e^{1/6}$$

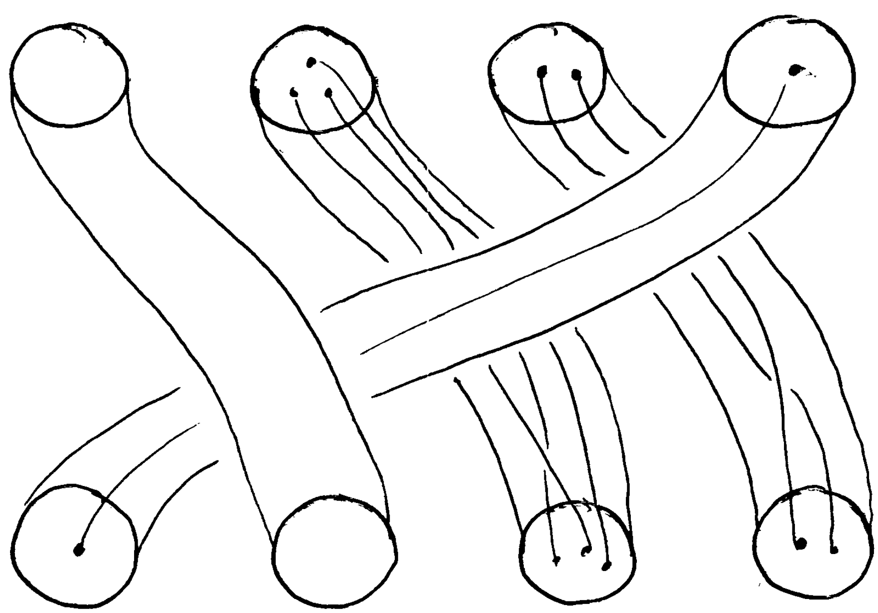
Now, how about:

$$E^{(E^{Z_0})} = \text{"the groupoid of finite sets labelled by } Z_0\text{-colored finite sets."}$$

or if $Z_0 = 1$:

$$E^{E^1} = E^E = \text{"the groupoid of finite sets labelled by finite sets."}$$

A morphism



objects here are "finite set bundles over finite sets".

The cardinality of this groupoid is

$$|E^E| = e^e \approx 15.154$$