The Riemann Zeta Function

John C. Baez, December 21, 2003

As we've seen in a previous homework, the attempt to compute finite sums by inverting the difference operator inevitably led Jacob Bernoulli to consider the Taylor series of a curious function:

$$\frac{z}{e^z - 1} = \sum_{k \geq 0} B_k \frac{z^k}{k!},$$

whose coefficients B_k *are now called the* **Bernoulli numbers**. *He used this technology to compute finite sums like*

$$1^p + \cdots + n^p.$$

However, he was stumped by infinite sums like

$$1^{-2} + 2^{-2} + 3^{-2} + \cdots$$

and this led him to pose the famous **Basel problem***: namely, to compute this sum!*

In 1735, a young fellow named Euler stunned the mathematical world by cracking the Basel problem and showing

$$1^{-2} + 2^{-2} + 3^{-2} + \cdots = \frac{\pi^2}{6}.$$

His approach illustrates the brilliant ruthlessness for which he was to become famous. He knew that whenever P *is a polynomial of degree* n *with zeroes at* $z_1, \ldots, z_n$*, we have*

$$P(z) = c\left(1 - \frac{z}{z_1}\right) \cdots \left(1 - \frac{z}{z_n}\right).$$

Although the function

$$P(z) = \frac{\sin z}{z}$$

is not a polynomial, he guessed that a similar formula would apply. Since this function has zeroes at $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$*, this would mean*

$$\begin{aligned}
\frac{\sin z}{z} &= c\left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\left(1 + \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{3\pi}\right)\cdots \\
&= c\left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{(2\pi)^2}\right)\left(1 - \frac{z^2}{(3\pi)^2}\right)\cdots
\end{aligned}$$

Since the left side goes to 1 *as* $z \to 0$*, the constant* c *must equal* 1*:*

$$\frac{\sin z}{z} = \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{(2\pi)^2}\right)\left(1 - \frac{z^2}{(3\pi)^2}\right)\cdots$$

Despite the wacky way Euler got it, this formula is true: the product converges to the right answer for all $z \in \mathbb{C}$*. On the other hand, Euler knew*

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

which gives another formula for $\sin z/z$*. Equating these two, he obtained*

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{(2\pi)^2}\right)\left(1 - \frac{z^2}{(3\pi)^2}\right)\cdots$$

Multiplying out the right-hand side and equating the coefficient of z^2 on both sides, he got:

$$-\frac{1}{3!} = -\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - \cdots$$

or

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Voilà!

As he admitted,

"The method was new and never used yet for such a purpose."

To check it, he computed the sum by hand to seven decimal places and saw that it agreed with $\pi^2/6$ to this accuracy.

One year later, he went further and computed the sum

$$\zeta(s) = \sum_{n \geq 1} n^{-s},$$

for all positive even integers s. Not surprisingly, computing a sum of powers like this involves the Bernoulli numbers. Let's see how he did it!

1. Show that

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2}\coth(\frac{z}{2})$$

where coth is a rather odd function, the **hyperbolic cotangent**:

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

2. Using part 1 and the definition of the Bernoulli numbers, show that $B_n = 0$ when n is odd and greater than 1.

In fact we have:

$$\begin{array}{rclrcl}
B_0 & = & 1 & B_1 & = & -\frac{1}{2} \\
B_2 & = & \frac{1}{6} & B_3 & = & 0 \\
B_4 & = & -\frac{1}{30} & B_5 & = & 0 \\
B_6 & = & \frac{1}{42} & B_7 & = & 0 \\
B_8 & = & -\frac{1}{30} & B_9 & = & 0 \\
B_{10} & = & \frac{5}{66} & B_{11} & = & 0 \\
B_{12} & = & -\frac{691}{2730} & B_{13} & = & 0 \\
B_{14} & = & \frac{7}{6} & B_{15} & = & 0 \\
B_{16} & = & -\frac{3617}{510} & B_{17} & = & 0
\end{array}$$

and the even Bernoulli numbers keep getting bigger *in absolute value from this point on! In fact they grow very rapidly:*

$$|B_{2k}| \sim 4\left(\frac{k}{\pi e}\right)^{2k}\sqrt{\pi k}.$$

3. Using parts 1 and 2, show that as formal power series

$$z \cot z = \sum_{k \geq 0} B_{2k} \frac{(2iz)^{2k}}{(2k)!}$$

It follows that they agree as functions within the radius of convergence of the right-hand side, which is π, since the poles of $z \cot z$ closest to the origin occur at $\pm\pi$.

Now, to get something interesting out of this, we'll use another cool formula:

$$\sum_{n=-\infty}^{\infty} \frac{1}{z-n} = \pi \cot \pi z.$$

where since the sum isn't absolutely convergent, we must be careful to sum from $n = -N$ to $n = +N$ and then take the limit $N \to +\infty$. I'm too lazy to get you to prove this, but it should at least be mildly plausible, given that both sides have simple poles at all integers and vanish at all half-integers.

4. Starting from the above formula, first show that

$$z \cot z = 1 - 2 \sum_{n \geq 1} \frac{z^2}{n^2\pi^2 - z^2},$$

and then, writing each term as a geometric series, show that

$$z \cot z = 1 - 2 \sum_{k \geq 1} \zeta(2k) \left(\frac{z}{\pi}\right)^{2k}.$$

5. Comparing the formulas in parts 3 and 4, find an explicit formula for $\zeta(2k)$ in terms of the Bernoulli number B_{2k}.

6. To see just how cool this is, work out

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

and

$$\zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

and

$$\zeta(6) = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots$$

7. What structure type would seem to have the generating function

$$\frac{z}{1-e^z} = -\sum_{n \geq 0} B_n \frac{z^n}{n!} \ ?$$

What is the problem with this idea?

The function ζ is now called the **Riemann zeta function**, *since mathematical discoveries are never named after the people who made them. Bernhard Riemann did, however, develop the theory* *of analytic continuation needed to rigorously define* $\zeta(s)$ *for all* $s \in \mathbb{C} - \{1\}$*. This allows us to justify Euler's far-out 'proof' that* $\zeta(-1) = -1/12$*, and prove the general formula*

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

for $n = 1, 2, 3, \ldots$*. Note this implies* $\zeta(-n) = 0$ *when* n *is even and greater than* 0*. These are called the* **trivial zeros** *of the zeta function.*

But Riemann didn't stop there: in 1859, starting with Euler's factorization of the zeta function:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

he derived an explicit formula for the prime numbers in terms of the zeros of the zeta function! He also posed the **Riemann Hypothesis***: if* $\zeta(z) = 0$*, then either* z *is a trivial zero or* z *lies on the* **critical line** $\mathrm{Re}(z) = \frac{1}{2}$*. This hypothesis is equivalent to the following estimate: the number* $\pi(x)$ *of prime numbers* $\leq x$ *satisfies*

$$|\pi(x) - \int_2^x \frac{dt}{\ln t}| \leq c\sqrt{x}\,\ln x$$

for some constant c*.*

It has been proved by now that all the nontrivial zeros lie in the **critical strip** $0 < \mathrm{Re}(z) < 1$*, and that there are infinitely many of them. The ZetaGrid project has used a large network of computers to show that the first* $250{,}000{,}000{,}000$ *zeroes in the critical strip lie on the critical line, and their calculations continue as you read this. However, a general proof remains out of reach. The Riemann Hypothesis is part of a larger constellation of conjectures and theorems relating algebraic number theory to 'zeta functions'. But, experts say that none of this network of ideas offers a plausible strategy to prove the Hypothesis! Hugh Montgomery writes:*

> "Sometimes I think that we essentially have a complete proof of the Riemann Hypothesis except for a gap. The problem is, the gap occurs right at the beginning, and so it's hard to fill that gap because you don't see what's on the other side of it."

One promising clue is that the distribution of Riemann zeroes on the critical line resembles the distribution of eigenvalues of a large $n \times n$ *hermitian matrix whose entries are randomly chosen using independent Gaussian probability distributions. Such random matrices also show up in the study of 'quantum chaos', so many people believe the key to proving the Riemann Hypothesis will be the discovery of a chaotic quantum system related to the prime numbers, whose Hamiltonian has eigenvalues given by the imaginary parts of the nontrivial zeta zeros. Many properties of this hoped-for system can be inferred from facts about the Riemann zeta function, but nobody has found it yet!*

8. *Extra Credit:* Prove the Riemann Hypothesis and win $1,000,000 from the Clay Mathematics Institute.

Hint: if you don't have any good ideas, join the ZetaGrid project at http://www.zetagrid.net/ *and let your computer run a screensaver that searches for Riemann zeta zeros. They offer a number of prizes.*