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Quantum Gravity Seminar
Homework #7

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Riemann ζ-Function

1) $$\frac{z}{e^z-1} + \frac{z}{2} = \frac{2z + z(e^z-1)}{2(e^z-1)}$$

$$= \frac{ze^z+z}{2(e^z-1)}$$

$$= \frac{z}{2}\cdot\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}}.$$ (Mult. by $1 = \frac{e^{-z/2}}{e^{-z/2}}$)

$$= \frac{z}{2}\coth\frac{z}{2}$$

2) As John noted, coth is a rather odd function, which makes $\frac{z}{2}\coth\frac{z}{2}$ a rather even function. Hence all of the odd terms in the Taylor expansion of $\frac{z}{2}\coth\frac{z}{2}$ must be zero. Using $\frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n$,

$$\frac{z}{2}\coth\frac{z}{2} = \frac{z}{2} + \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n$$

$$= B_0 + \left(B_1+\frac{1}{2}\right)z + \sum_{n=2}^{\infty}\frac{B_n}{n!}z^n,$$

whence we obtain:

$$B_1 = \frac{-1}{2}$$ (already seen in a previous homework)

& $B_{2n+1} = 0 \quad \forall n \geq 1$.

Now we can write more simply:

$$\frac{z}{2}\coth\frac{z}{2} = \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n}$$

✓

3) $$z \cot z = z\frac{\cos z}{\sin z}$$
$$= z\frac{\cosh iz}{-i\sinh iz}$$
$$= iz \coth iz$$
$$= \frac{2iz}{2}\coth\frac{2iz}{2}$$
$$= \sum_{k=0}^{\infty}\frac{B_{2k}}{(2k)!}(2iz)^{2k} \qquad \text{(by part 2)}$$

4) Since $\pi \cot \pi z = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z-n}$, we have:

$$z\cot z = \frac{z}{\pi}\left(\pi\cot\pi\left(\frac{z}{\pi}\right)\right)$$
$$= \frac{z}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{\left(\frac{z}{\pi}\right)-n}$$
$$= \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{z}{z-n\pi}$$
$$= \lim_{N\to\infty}\left[1+\sum_{n=1}^{N}\left(\frac{z}{z-n\pi}+\frac{z}{z+n\pi}\right)\right]$$
$$= 1+\sum_{n=1}^{\infty}\frac{z(z+n\pi)+z(z-n\pi)}{z^2-n^2\pi^2}$$
$$= 1+\sum_{n=1}^{\infty}\frac{2z^2}{z^2-n^2\pi^2}$$
$$= \boxed{1-2\sum_{n=1}^{\infty}\frac{z^2}{n^2\pi^2-z^2}}$$
$$= 1-2\sum_{n=1}^{\infty}\frac{\frac{z^2}{n^2\pi^2}}{1-\frac{z^2}{n^2\pi^2}}$$
$$= 1-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{z^2}{n^2\pi^2}\right)^k$$
$$= 1-2\sum_{k=1}^{\infty}\left(\frac{z}{\pi}\right)^{2k}\sum_{n=1}^{\infty}n^{-2k} = \boxed{1-2\sum_{k=1}^{\infty}\left(\frac{z}{\pi}\right)^{2k}\zeta(2k)}$$

5)

$$z\cot z = z\cot z$$

$$(\text{PART } 3) \rightarrow \quad \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!}(2iz)^{2k} = 1 - 2\sum_{k=1}^{\infty}\left(\frac{z}{\pi}\right)^{2k}\zeta(2k) \quad \leftarrow (\text{PART } 4)$$

Equating coefficients, we see that for $k \geq 1$,

$$\frac{B_{2k}}{(2k)!}(2i)^{2k} = \frac{-2\zeta(2k)}{\pi^{2k}}$$

or

$$\zeta(2k) = \frac{-\pi^{2k}(2i)^{2k}B_{2k}}{2(2k)!}$$

$$\boxed{\zeta(2k) = \frac{(2\pi)^{2k}(-1)^{k+1}B_{2k}}{2(2k)!}}$$

✓

6) $$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{(2\pi)^2(-1)^2\left(\frac{1}{6}\right)}{2\cdot 2!} = \boxed{\frac{\pi^2}{6}}$$

$$\zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{(2\pi)^4(-1)^3\left(-\frac{1}{30}\right)}{2\cdot 4!} = \frac{16\pi^4}{48\cdot 30} = \boxed{\frac{\pi^4}{90}}$$

$$\zeta(6) = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{(2\pi)^6(-1)^4\left(\frac{1}{42}\right)}{2\cdot 6!} = \frac{64\pi^6}{60480} = \boxed{\frac{\pi^6}{945}}$$

✓

7) What structure type would seem to have the generating function $\frac{z}{1-e^z} = -\sum_{n\geq 0} B_n \frac{z^n}{n!}$? Why, $\frac{Z}{1-E^Z}$ of course! But what does this mean? First let's figure out $\frac{1}{1-E^Z}$. This is the composition of two structure types:

GOOD!

E^Z — the vacuous structure

$\frac{1}{1-Z}$ — total ordering

So to put a $\frac{1}{1-E^Z}$-structure on S, we break S up into disjoint subsets, put the vacuous structure on each subset and totally order the set of subsets. That is, we split S into disjoint subsets, ordering the set of subsets but imposing no structure on their elements. Now we'd like to define $\frac{Z}{1-E^Z} = Z \cdot \frac{1}{1-E^Z}$, using the usual product of structure types, so that $\frac{Z}{1-E^Z}$-structuring a set S means splitting off a 1-element subset of S and putting a $\frac{1}{1-E^Z}$-structure on $S-1$.

The problem with this idea? Well, doggone it if the number of ways of putting a "$\frac{Z}{1-E^Z}$"-structure on n doesn't always seem to turn out to be a <u>nonnegative integer</u>! Since $-B_n$ is generally fractional and is negative as often as positive, this is a problem.

Ooops! good point!

no! it's infinite!

There are ∞ly many ways to chop up S into possibly empty sets.