QUANTUM GRAVITY HOMEWORK 7

ERIN PEARSE

1. Since $\coth\frac{z}{2} = \frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}}$, we get

$$\begin{aligned}
\frac{z}{2}\coth\frac{z}{2} &= \frac{z}{2}\left(\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}}\right) \\
&= \frac{z}{2}\cdot\frac{e^{z/2}}{e^{z/2}}\left(\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}}\right) \\
&= \frac{z}{2}\cdot\frac{e^{z}+e^{0}}{e^{z}-e^{0}} \\
&= \frac{z(e^z+1)}{2(e^z-1)} \\
&= \frac{ze^z+z+(z-z)}{2(e^z-1)} \\
&= \frac{2z}{2(e^z-1)}+\frac{z(e^z-1)}{2(e^z-1)} \\
&= \frac{z}{e^z-1}+\frac{z}{2}
\end{aligned}$$

2. Since the Bernoulli numbers are defined by the relation

$$\frac{z}{e^z-1} = \sum_{k\geq 0} B_k\frac{z^k}{k!},$$

we obtain

$$\frac{z}{2}+\frac{z}{e^z-1} = \frac{z}{2}+\sum_{k\geq 0} B_k\frac{z^k}{k!} = \frac{z}{2}\coth\frac{z}{2} \tag{1}$$

by problem 1. Since $\coth z$ is an odd function, $z\coth z$ is an even function and hence so is $\frac{z}{2}\coth\frac{z}{2}$. It is the defining property of even functions that their power series expansion has no odd-powered terms, or rather, that the coefficient of z^{2n+1} is 0 for every n. Then collecting coefficients, the second equality in (1) can be rewritten

$$B_0+\left(\frac{1}{2}+B_1\right)z+\sum_{k\geq 1} B_{2k}\frac{z^{2k}}{(2k)!} = \frac{z}{2}\coth\frac{z}{2},$$

which immediately implies

$$B_1 = -\frac{1}{2}, \quad \text{and } B_{2n+1} = 0,\ \forall n \geq 1.$$

3. Now we obtain a power series:

$$\begin{aligned}
\frac{z}{2}\coth\frac{z}{2} &= \frac{z}{2} + \frac{z}{e^z - 1} \\
&= \frac{z}{2} + \sum_{k\geq 0} B_k \frac{z^k}{k!} \\
&= \frac{z}{2} + \left(B_0 + B_1 z + B_2\frac{z^2}{2!} + B_3\frac{z^2}{3!} + B_4\frac{z^4}{4!} + B_5\frac{z^5}{5!} + B_6\frac{z^6}{6!} + \dots\right) \\
&= \frac{z}{2} + \left(B_0 - \frac{1}{2}z + B_2\frac{z^2}{2!} + (0)\frac{z^2}{3!} + B_4\frac{z^4}{4!} + (0)\cdot\frac{z^5}{5!} + B_6\frac{z^6}{6!} + \dots\right) \qquad (2) \\
&= B_0 + B_2\frac{z^2}{2!} + B_4\frac{z^4}{4!} + B_6\frac{z^6}{6!} + \dots \\
&= \sum_{k\geq 0} B_{2k}\frac{z^{2k}}{(2k)!}
\end{aligned}$$

In (2), the odd terms (after the first) drop out by the result in problem 2. Now making the substitution $\frac{z}{2} \mapsto z$,

$$\frac{z}{2}\coth\frac{z}{2} = \sum_{k\geq 0} B_{2k}\frac{z^{2k}}{(2k)!} \qquad \Longrightarrow \qquad z\coth z = \sum_{k\geq 0} B_{2k}\frac{(2z)^{2k}}{(2k)!}.$$

Then making another substitution $z \mapsto iz$,

$$iz\coth iz = \sum_{k\geq 0} B_{2k}\frac{(2iz)^{2k}}{(2k)!}.$$

But note that we can go another way with this:

$$\begin{aligned}
\sum_{k\geq 0} B_{2k}\frac{(2iz)^{2k}}{(2k)!} &= iz\coth iz \\
&= iz\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} && \text{def of } \coth z \\
&= iz\frac{e^{iz}+e^{-iz}}{2}\cdot\frac{2i}{e^{iz}-e^{-iz}}\cdot\frac{1}{i} \\
&= iz\frac{\cos z}{\sin z}(-i) && \text{def of } \cos z, \sin z \\
&= z\cot z,
\end{aligned}$$

and we are done.

4. Since

$$\pi \cot \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{z-n},$$

we make the substitution $\pi z \mapsto z$ to obtain

$$\pi \cot z = \sum_{n=-\infty}^{\infty} \frac{1}{z/\pi - n}.$$

Then, multiply both sides by z/π to obtain

$$\begin{aligned}
z \cot z &= \frac{z}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{z/\pi - n} \\
&= \sum_{n=-\infty}^{\infty} \frac{z}{\pi} \cdot \frac{1}{z/\pi - n} \\
&= \sum_{n=-\infty}^{\infty} \frac{z}{z - n\pi} \frac{z + n\pi}{z + n\pi} \\
&= \sum_{n=-\infty}^{\infty} \frac{z^2 + zn\pi}{z^2 - n^2\pi^2} \\
&= \sum_{n=-\infty}^{\infty} \frac{z^2}{z^2 - n^2\pi^2} + \sum_{n=-\infty}^{\infty} \frac{zn\pi}{z^2 - n^2\pi^2} \\
&= \sum_{n=-\infty}^{\infty} \frac{z^2}{z^2 - n^2\pi^2} \qquad (3) \\
&= \sum_{n=1}^{\infty} \frac{z^2}{z^2 - (-n)^2\pi^2} + 1 + \sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2\pi^2} \\
&= 1 - 2\sum_{n=1}^{\infty} \frac{z^2}{n^2\pi^2 - z^2}
\end{aligned}$$

Note that the second sum vanishes in line (3) because

$$\sum_{n=-\infty}^{\infty} \frac{zn\pi}{z^2 - n^2\pi^2} = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{zn\pi}{z^2 - n^2\pi^2},$$

and the terms of this latter sum cancel pairwise:

$$\frac{zn\pi}{z^2 - n^2\pi^2} + \frac{z(-n)\pi}{z^2 - n^2\pi^2} = 0, \forall n.$$

Now using

$$z \cot z = 1 - 2\sum_{n\geq 1} \frac{z^2}{n^2\pi^2 - z^2}, \qquad (4)$$

we would like to show

$$z\cot z = 1 - 2\sum_{k\geq 1}\zeta(2k)\left(\frac{z}{\pi}\right)^k.$$

Working backwards just for kicks,

$$\begin{aligned}
\sum_{k\geq 1}\zeta(2k)\left(\frac{z}{\pi}\right)^{2k} &= \sum_{k\geq 1}\sum_{n\geq 1}\frac{1}{n^{2k}}\cdot\frac{z^{2k}}{\pi^{2k}} \\
&= \sum_{n\geq 1}\sum_{k\geq 1}\left(\frac{z^2}{n^2\pi^2}\right)^k \qquad (5)\\
&= \sum_{n\geq 1}\left(\frac{1}{1 - z^2/n^2\pi^2} - 1\right) \qquad (6)\\
&= \sum_{n\geq 1}\left(\frac{n^2\pi^2}{n^2\pi^2 - z^2} - \frac{n^2\pi^2 - z^2}{n^2\pi^2 - z^2}\right) \\
&= \sum_{n\geq 1}\frac{z^2}{n^2\pi^2 - z^2}. \qquad (7)
\end{aligned}$$

Note that the sums may be interchanged in (5) because the summands are all positive (since everything is squared), and that the -1 in (6) comes from the missing 0 term:

$$\frac{1}{1-r} = \sum_{n\geq 0} r^n = 1 + \sum_{n\geq 1} r^n.$$

We can use the geometric series formula here because we are restricting to the disk $|z| < \pi$, where the summand is

$$\left|\frac{z^2}{n^2\pi^2}\right| = \frac{z^2}{n^2\pi^2} < 1,\ \forall n \geq 1.$$

Finally, we substitute (7) back into (4) and obtain

$$z\cot z = 1 - 2\sum_{n\geq 1}\frac{z^2}{n^2\pi^2 - z^2} = 1 - 2\sum_{k\geq 1}\zeta(2k)\left(\frac{z}{\pi}\right)^k.$$

5. From 3&4 we obtain

$$\sum_{k\geq 0} B_{2k}\frac{(2iz)^{2k}}{(2k)!} = 1 - 2\sum_{k\geq 1}\zeta(2k)\left(\frac{z}{\pi}\right)^{2k}$$

$$\implies 1 + \sum_{k\geq 1} B_{2k}\frac{(2iz)^{2k}}{(2k)!} = 1 - \sum_{k\geq 1} 2\zeta(2k)\left(\frac{z}{\pi}\right)^{2k}.$$

Now we can collect terms and solve for $\zeta(2k)$:

$$\sum_{k\geq 1}\left(B_{2k}\frac{(2iz)^{2k}}{(2k)!}+2\zeta(2k)\left(\frac{z}{\pi}\right)^{2k}\right)=0$$

$$B_{2k}\frac{(2i)^{2k}}{(2k)!}+2\zeta(2k)\pi^{-2k}=0$$

$$2\zeta(2k)\pi^{-2k}=-B_{2k}\frac{(2i)^{2k}}{(2k)!}$$

$$\zeta(2k)=-\frac{\pi^{2k}}{2}B_{2k}\frac{(2i)^{2k}}{(2k)!}$$

6. Now we can compute (and this is cool!):

$$\begin{aligned}
\zeta(2) &= -\frac{\pi^2}{2}\left(\frac{1}{6}\right)\frac{(2i)^2}{2!}=\frac{\pi^2}{2}\left(\frac{1}{6}\right)\frac{2^2}{2}=\frac{\pi^2}{6}\\
\zeta(4) &= -\frac{\pi^4}{2}\left(-\frac{1}{30}\right)\frac{(2i)^4}{4!}=\frac{\pi^4}{2}\left(\frac{1}{30}\right)\frac{2^4}{2\cdot 3\cdot 2^2}=\frac{\pi^4}{90}\\
\zeta(6) &= -\frac{\pi^6}{2}\left(\frac{1}{42}\right)\frac{(2i)^6}{6!}=\frac{\pi^6}{2}\left(\frac{1}{2\cdot 21}\right)\frac{2^6}{2\cdot 3\cdot 2^2\cdot 5\cdot 2\cdot 3}=\frac{\pi^6}{945}\\
&\vdots
\end{aligned}$$

7. What structure type would seem to have the generating function

$$\frac{z}{1-e^z}=-\sum_{n\geq 0}B_n\frac{z^n}{n!}?$$

The structure type of "not being a simply-connected orbifold"?
No, really I have no idea. In any case, I think the trouble is that it's negative ...