ζ

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1. Starting from the definition of that rather odd function (or, rather, that odd function), the hyperbolic cotangent, we see

$$\begin{aligned}\frac{z}{2}\coth\left(\frac{z}{2}\right) &= \frac{z}{2}\cdot\frac{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\\ &= \frac{z}{2}\cdot\frac{e^{z}+1}{e^{z}-1}\\ &= \frac{z}{2}\cdot\left(\frac{2}{e^{z}-1}+1\right)\\ &= \frac{z}{e^{z}-1}+\frac{z}{2}\end{aligned}$$

2. The Bernoulli numbers are the coefficients of the power series for $\frac{z}{e^z-1}$, which are thus the coefficients for $\frac{z}{2}\coth\left(\frac{z}{2}\right)$ except for the coefficient of z^1, which is $\frac{1}{2}$ less in the series for $\frac{z}{e^z-1}$. But the hyperbolic cotangent is an odd function, all right, and so $\frac{z}{2}\coth\left(\frac{z}{2}\right)$ is an even function, and thus the coefficients of all odd powers of z in its Taylor expansion must vanish. So the only nonzero odd Bernoulli number is $B_1=-\frac{1}{2}$.

3. If we observe that $\cot(z)=\coth(iz)$ (since the corresponding fact holds for hyperbolic sine and cosine), so $z\cot(z)=z\coth(iz)$, and using part 1 with $2iz$ instead of z, we find that this is

$$\begin{aligned}z\cot(z) &= z\coth(iz)\\ &= \frac{2iz}{e^{2iz}-1}+2iz\\ &= \sum_{n\geqslant 0}B_n\frac{(2iz)^n}{n!}+iz\end{aligned}$$

Now, since $B_1=-\frac{1}{2}$, we see that the coefficient of z in the sum is just $-i$, which cancels the extra iz. Since all other odd Bernoulli numbers are zero, this leaves only the even coefficients, so we can let $n=2k$ to get

$$z\cot(z)=\sum_{k\geqslant 0}B_{2k}\frac{(2iz)^{2k}}{(2k)!}$$

4. From the fact that $\pi\cot(\pi z)=\sum_{n=-\infty}^{\infty}\frac{1}{z-n}$, we get that $z\cot(z)=(\frac{z}{\pi})\pi\cot(\pi\frac{z}{\pi})$ is given by

$$\begin{aligned}(\tfrac{z}{\pi})\pi\cot(\pi\tfrac{z}{\pi}) &= \frac{z}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\frac{z}{\pi}-n}\\ &= z\sum_{n=-\infty}^{\infty}\frac{1}{z-n\pi}\\ &= z\left[\frac{1}{z}+\sum_{n\geqslant 1}\left(\frac{1}{z-n\pi}+\frac{1}{z+n\pi}\right)\right] \quad ccl\\ &= 1+z\sum_{n\geqslant 1}\frac{2z}{z^2-n^2\pi^2}\\ &= 1-2\sum_{n\geqslant 1}\frac{z^2}{n^2\pi^2-z^2}\end{aligned}$$

Now for any given n, consider the term $\frac{z^2}{n^2\pi^2-z^2}$. We can express this as a geometric series scaled by a factor of $n^2\pi^2$:

$$\begin{aligned}
\frac{z^2}{n^2\pi^2-z^2} &= \frac{z^2}{n^2\pi^2}\cdot\frac{1}{1-\left(\frac{z}{n\pi}\right)^2}\\
&= \frac{z^2}{n^2\pi^2}\sum_{k\geqslant 0}\left(\frac{z}{n\pi}\right)^{2k}\\
&= \sum_{k\geqslant 0}\left(\frac{z}{\pi}\right)^{2(k+1)} n^{-2(k+1)}\\
&= \sum_{k\geqslant 1}\left(\frac{z}{\pi}\right)^{2k} n^{-2k}
\end{aligned}$$

Now summing over n, we get[1]

$$\begin{aligned}
z\cot(z) &= 1-2\sum_{n\geqslant 1}\frac{z^2}{n^2\pi^2-z^2}\\
&= 1-2\sum_{n\geqslant 1}\left(\sum_{k\geqslant 1} n^{-2k}\left(\frac{z}{\pi}\right)^{2k}\right)\\
&= 1-2\sum_{k\geqslant 1}\left(\sum_{n\geqslant 1} n^{-2k}\right)\left(\frac{z}{\pi}\right)^{2k}\\
&= 1-2\sum_{k\geqslant 1}\zeta(2k)\left(\frac{z}{\pi}\right)^{2k}
\end{aligned}$$

5. We now have two formulae for $z\cot(z)$, so if we equate them we find (using first the fact that $B_0 = 1$) that:

$$\begin{aligned}
z\cot(z) &= \sum_{k\geqslant 0} B_{2k}\frac{(2iz)^{2k}}{(2k)!}\\
&= 1+\sum_{k\geqslant 1} B_{2k}\frac{(2i)^{2k}}{(2k)!}z^{2k}\\
&= 1+\sum_{k\geqslant 1} B_{2k}\frac{2^{2k}(-1)^k}{(2k)!}\pi^{2k}\left(\frac{z}{\pi}\right)^{2k}\\
&= 1-2\sum_{k\geqslant 1}(-1)^{k-1}B_{2k}\frac{2^{2k-1}}{(2k)!}\pi^{2k}\left(\frac{z}{\pi}\right)^{2k}
\end{aligned}$$

and so, from the form found in part 4, we get that

$$1-2\sum_{k\geqslant 1} B_{2k}\frac{2^{2k-1}(-1)^{k-1}}{(2k)!}\pi^{2k}\left(\frac{z}{\pi}\right)^{2k} = 1-2\sum_{k\geqslant 1}\zeta(2k)\left(\frac{z}{\pi}\right)^{2k}$$

This means, equating coefficients, that

$$\zeta(2k) = (-1)^{k-1}B_{2k}\frac{2^{2k-1}}{(2k)!}\pi^{2k}$$

6. So now we have an explicit formula for $\zeta(2k) = \sum_{n\geqslant 1}\frac{1}{n^{2k}}$ with which we can find some special examples:

$$\begin{aligned}
\zeta(2) &= (-1)^0 B_2\frac{2^1}{2!}\pi^2\\
&= (1)\left(\frac{1}{6}\right)\left(\frac{2}{2}\right)\pi^2\\
&= \frac{\pi^2}{6}\\
\zeta(4) &= (-1)^1 B_4\frac{2^3}{4!}\pi^4\\
&= -\left(-\frac{1}{30}\right)\left(\frac{8}{24}\right)\pi^4\\
&= \frac{\pi^4}{90}\\
\zeta(6) &= (-1)^2 B_6\frac{2^5}{6!}\pi^6\\
&= \left(\frac{1}{42}\right)\left(\frac{32}{720}\right)\pi^6\\
&= \frac{\pi^6}{945}
\end{aligned}$$

[1]Note that this is all without reference to questions of convergence, so we needn't worry about exchanging the sums.

Groovy.

7. The function $\frac{z}{1-e^z}$ would appear to be the generating function of a structure type we could write $\frac{Z}{1-E^Z}$, which, applied to a set S, decomposes S into two parts, one of which is given a Z structure (i.e. the structure of being a one-element set), and the other given a structure of type $\frac{1}{1-E^Z}$, which would be[2] the structure of being an ordered set of finite (possibly empty) sets[3].
One problem with this description is that such a structure can be put on the empty set in an infinite number of ways (one could take an ordered set of any number of copies of the empty set, and this would be a $\frac{1}{1-E^Z}$-structure on the empty set). This problem corresponds to the fact that the generating function of this purported structure type would have to be $(1-e^z)^{-1}$, which has a pole at 0. Since the generating function can be seen as a Taylor series expanded about $z = 0$, this is a problem (not only does the series not converge anywhere, but the coefficients are not even well-defined natural numbers!). So even though the function $\frac{z}{1-e^z}$ is analytic and converges in some neighborhood of 0 (everywhere, in fact), it cannot be decomposed as a product of analytic functions which converge near 0 - this prevents its definition as a structure type in the usual way.
Another problem which suggests that this function may be a problem is the fact that the coefficients B_n are not all nonnegative, and are not even all integers (in general we need to use rational numbers for the B_n), so to interpret them as cardinalities of a set (or other non-set entity such as a groupoid) of structures takes us outside the category of structure types.

8. *(Exercise)*

[2]This description relies on a characterization of composition of structure types, namely that given two structure types F and G, a $G \circ F$ structure on a set S is a way of breaking S into parts, putting F structures on the parts, and putting a G structure on the set of parts. We haven't discussed this in seminar, but I contend it is obvious enough from looking at composition of power series carefully that this isn't really necessary.

[3]Note that this is slightly different from the type which Bergeron, Labelle and Leroux call a "ballot", which is an ordered partition, or, as we would put it, $\frac{1}{1-(E^Z-1)}$ (in their terminology, $L(E_+)$, rather than the example from the question, which they would call $L(E)$ if it made sense). This is an ordered set of NONEMPTY finite sets - the name "ballot" coming from the fact that this sort of structure is what each voter puts on sets of candidates in a certain class of (nondegenerate) social-choice algorithms. Which is not really significant to answering the question, but does provide some context and significance to the remark that this structure lacks the serious flaw of the example in this question.