Euler's Crazy Proof

1. A k-pointing of n is a function \( k \rightarrow n \), and there are \( n^k \) of these. There are \( n! \) orderings of \( n \), so if a \( P(k) \)-str. on \( n \) is an ordering and k-pointing of \( n \), then

\[
|P(k)| = n! \cdot n^k.
\]

2. \( |P(k)| (z) = \sum_{n=0}^{\infty} \frac{n! \cdot n^k \cdot z^n}{n!} = \sum_{n=0}^{\infty} n^k \cdot z^n. \]

3. \( |P(0)| (z) = \sum_{n=0}^{\infty} n^0 \cdot z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}. \)

4. An \( N\Psi \)-str. on \( S \), i.e. an \( A^*A\Psi \)-str. on \( S \), is:

A choice of \( x \in S \) and an \( A\Psi \)-str. on \( S-\{x\} \).

i.e.: A choice of \( x \in S \) and a \( \Psi \)-str. on \((S-\{x\})+1\)

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(Note: if we consider instead an \( AA^*\Psi \)-str. on \( S \), we see that this amounts to a choice of \( x \in S+1 \) and a \( \Psi \)-str. on \( S \). Then we can interpret an \([A,A^*]\Psi \)-str. on \( S \) as a choice of any \( x \in S+1 \) such that \( x \in S \), and then a \( \Psi \)-str. on \( S \). Since the "choice" of \( x \) is really no choice at all, this is the same as just a \( \Psi \)-str. on \( S \). This gives us \([A,A^*] \equiv 1 \) as an isomorphism between operators on structure types.)
5) An NP(k)-structure on S is a choice of \( x \in S \), an ordering of S and a k-pointing \( k \rightarrow S \). But a k-pointing \( k \rightarrow S \) together with a choice of \( x \in S \) is the same as a \( k+1 \) pointing \( k+1 \rightarrow S \), where the additional element is mapped to \( x \). Conversely, given \( k+1 \rightarrow S \) and an ordering of S, we can call the image of the last point of S (according to the order) a "choice of \( x \in S \)" and get a k-pointing by restricting the domain of \( k+1 \rightarrow S \) to \( k \).

This obviously gives an isomorphism of structure types:

\[ \text{NP}(k) \cong P(k+1). \]

6) Decategorifying, we get

\[ |\text{A}^k A \{P(k)\}|(z) = |P(k+1)|\{z\} \]

\[ 0^*z |P(k)|\{z\} = |P(k+1)|\{z\} \]

\[ z \frac{d}{dz} |P(k)|\{z\} = |P(k+1)|\{z\} \]

Explicitly, this just says:

\[ z \frac{d}{dz} \sum_{n=0}^{\infty} n^k z^n = \sum_{n=0}^{\infty} n^{k+1} z^n \]

\[ z \sum_{n=0}^{\infty} n^{k+1} z^{n-1} = \sum_{n=0}^{\infty} n^{k+1} z^n \]

or

\[ \sum_{n=0}^{\infty} n^{k+1} z^n = \sum_{n=0}^{\infty} n^{k+1} z^n \]

which is true!
7) \[ |P(1)|(z) = \frac{d}{dz} |P(0)|(z) \]

\[ = z \frac{d}{dz} \frac{1}{1 - z} \]

\[ = z \cdot \frac{1}{(1 - z)^2} \]

\[ = \frac{z}{(1 - z)^2}. \]

8) If we could evaluate \( |P(1)|(-1) \) at \(-1\), we would get:

\[ |P(1)|(-1) = \sum_{n=0}^{\infty} n(-1)^n = 0 -1 + 2 - 3 + 4 - 5 + \cdots \]

But also:

\[ |P(1)|(-1) = \frac{(-1)}{(1-(-1))^2} = \frac{-1}{4}, \text{ by \#7.} \]

So:

\[ "1 - 2 + 3 - 4 + 5 - 6 + \cdots = \frac{1}{4} " \]

9) \[ \sum_{n=1}^{\infty} (-1)^{n+1} n = \frac{1}{4}. \]

**Proof:** The Taylor expansion of \( \frac{z}{(1-z)^2} \) about \( z = 0 \) agrees with the formal power series \( |P(1)|(z) \), and converges for \( |z| < 1 \). In particular, we have

\[ \frac{-t}{(1-t)^2} = |P(1)|(-t) = \sum_{n=1}^{\infty} n(-t)^n \]

for \( t \in (0,1) \). The left-hand side of this has limit
\[
\frac{-1}{(1-(-1))^2} = \frac{-1}{4} \text{ as } \theta \to 1. \quad \text{For the right hand side, we get:}
\]

\[
\lim_{\theta \to 1} \sum_{n=1}^{\infty} n (-t)^n = -\lim_{\theta \to 1} \sum_{n=1}^{\infty} t^n (-1)^n n
\]

\[
= -A \sum_{n=1}^{\infty} (-1)^n n^2,
\]

by definition of the Abel sum. Hence \( A \sum_{n=1}^{\infty} (-1)^n n^2 = \frac{1}{4} \). □

10) \[|P(2)(z)| = z \frac{d}{dz} |P(1)(z)|
\]

\[
= z \frac{d}{dz} \frac{z}{(1-z)^2}
\]

\[
= z \left( \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3} \right)
\]

\[
= z \frac{1 + z}{(1-z)^3}
\]

Since this function is analytic in the disc \(|z| < 1\), we have, for all points \(z\) in the disc:

\[
\frac{z(1+z)}{(1-z)^3} = \sum_{n=1}^{\infty} n^2 z^n
\]

In particular, if \(t \in [0,1)\),

\[
-\frac{t(1+t)}{(1-t)^3} = -\sum_{n=1}^{\infty} n^2 t^n
\]

and taking the limit as \(t \to 1\), we find

\[
0 = A \sum_{n=1}^{\infty} (-1)^n n^2
\]

so the alternating (Abel) sum of the squares of all positive integers is zero!
Finally, we use this to find the ‘sum’ of all the squared positive integers:

\[ 1^2 + 2^2 + 3^2 + 4^2 + \cdots = \underbrace{(1^2 - 2^2 + 3^2 - 4^2 + \cdots)}_{\text{0}} + (2 \cdot 2^2 + 2 \cdot 4^2 + 2 \cdot 6^2 + \cdots) = \]

\[ = 2^3 \left( 1^2 + 2^2 + 3^2 + 4^2 + \cdots \right) \]

or

\[ (1 - 2^3)(1^2 + 2^2 + 3^2 + 4^2 + \cdots) = \Omega \]

\( \Rightarrow \quad \boxed{(1^2 + 2^2 + 3^2 + 4^2 + \cdots) = \Omega} \)

and indeed, the Riemann zeta function has zeros at all of the negative even integers!