QUANTUM GRAVITY HOMEWORK 5

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1. To put a $P(k)$-structure on the n-element set S, we first have to totally order the elements. There are $n!$ ways to do this. Then, each of the numbers $j \in \{1, 2, \dots, k\}$ must be assigned to one of the n elements of S. Since repeats are allowed (more than one j may be assigned to a given point of S), there are n possibilities for each j. Hence,
$$|P(k)_n| = n!n^k.$$

2. $|P(k)|(z) = \displaystyle\sum_{n\geq 0} \frac{|P(k)_n|}{n!} z^n = \sum_{n\geq 0} n^k z^n.$

3. $|P(0)|(z) = \displaystyle\sum_{n\geq 0} n^0 z^n = \sum_{n\geq 0} z^n = \frac{1}{1-z}.$

4. Define the number operator by $N\Psi = A^*A\Psi$. Then
$$\begin{aligned}
&\text{put a } N\Psi\text{-structure on a set } S \\
&\qquad = \text{put an } A^*A\Psi\text{-structure on } S \\
&\qquad \cong \text{choose } x \in S \text{ and put an } A\Psi\text{-structure on } S\backslash\{x\} \\
&\qquad \cong \text{choose } x \in S \text{ and put } \Psi\text{-structure on } (S\backslash\{x\}) + 1
\end{aligned}$$

   This corresponds to 'pointing' S (i.e., giving it one distinguished point) and putting a Ψ-structure on it.

5. Putting an $NP(k)$ structure on S amounts to pointing S, and then totally ordering S and giving it a k-pointing. But this is the same as totally ordering S, giving it a distinguished point (say we give one point a star), and then k-pointing S, because surely the order doesn't matter. Finally, by 'reindexing'
$$\{*\} \cup \{, 1, 2, \dots, k\} = \{*, 1, 2, \dots, k\} \cong \{1, 2, \dots, k+1\}$$
so that we've really just put a total ordering on S and then given it a $(k+1)$-pointing, i.e., put a $P(k+1)$-structure on it. Hence,
$$NP(k) \cong P(k+1).$$

6. Decategorifying $P(k+1) \cong NP(k)$,

$$\begin{aligned}
|P(k+1)|(z) &= |NP(k)|(z) \\
&= |A^*AP(k)|(z) && \text{def of } N \\
&= z\tfrac{d}{dz}|P(k)|(z) && |A^*| = z, |A| = \tfrac{d}{dz}
\end{aligned}$$

7. By the previous problem we know

$$\begin{aligned}
|P(1)|(z) &= z\tfrac{d}{dz}|P(0)|(z) \\
&= z\tfrac{d}{dz}\left(\tfrac{1}{1-z}\right) \\
&= \tfrac{z}{(1-z)^2}
\end{aligned}$$

8. By the previous formula,

$$|P(1)|(-1) = \tfrac{-1}{(1+1)^2} = -\tfrac{1}{4}.$$

But since $|P(1)|(z) = \sum_{n\geq 0} nz^n$ by the definition of $|P(k)|$ found in 3, so

$$|P(1)|(-1) = 1 - 2 + 3 - 4 + \ldots = -\tfrac{1}{4}.$$

9. Using the definition of Abel summation:

$$A\sum_n an := \lim_{t\uparrow 1} \sum_n t^n a_n,$$

we see that

$$\begin{aligned}
A\sum_{n=1}^{\infty}(-1)^{n+1}n &= \lim_{t\uparrow 1}\sum_{n=1}^{\infty} t^n(-1)^{n+1}n && \text{by def} \\
&= -\lim_{t\uparrow 1}\sum_{n=1}^{\infty}(-t)^n n && \text{collecting} \\
&= -\lim_{t\uparrow 1}|P(1)|(-t) && \text{by 2} \\
&= -\lim_{t\uparrow 1}\tfrac{-t}{(1+t)^2} && \text{by 7} \\
&= -\left(\tfrac{-1}{2^2}\right) && \text{continuity} \\
&= \tfrac{1}{4}
\end{aligned}$$

10. First we compute:

$$\begin{aligned}|P(2)|(z) &= z\tfrac{d}{dz}|P(1)|(z)\\ &= z\tfrac{d}{dz}\tfrac{z}{(1-z)^2}\\ &= z\left(\tfrac{1}{(1-z)^2}+\tfrac{2z}{(1-z)^3}\right)\\ &= \tfrac{z}{(1-z)^2}+\tfrac{2z^2}{(1-z)^3}\end{aligned}$$

Now, to compute the Abel sum of $1^2-2^2+3^2-4^2+\dots$.

$$\begin{aligned}|P(2)|(-1) &= A\sum_{n=1}^{\infty} n^2(-1)^n\\ &= \lim_{t\uparrow 1}\sum_{n=1}^{\infty} t^n n^2(-1)^n\\ &= \lim_{t\uparrow 1}\sum_{n=1}^{\infty} n^2(-t)^n\\ &= \lim_{t\uparrow 1}|P(2)|(-t)\\ &= \lim_{t\uparrow 1}\tfrac{-t}{(1+t)^2}+\tfrac{2t^2}{(1+t)^3}\\ &= \tfrac{-1}{4}+\tfrac{2}{8}\\ &= 0\end{aligned}$$

Lastly, we wish to find $\zeta(-2)=1^2+2^2+3^2+4^2+\dots$. From

$$\zeta(-2)=\sum_{n=1}^{\infty} n^2 = 1^2+2^2+3^2+4^2+\dots,$$

we get

$$2^2\zeta(-2)=2^2\sum_{n=1}^{\infty} n^2=\sum_{n=1}^{\infty}(2n)^2=2^2+4^2+6^2+8^2+\dots.$$

Subtracting,

$$\zeta(-2)-2\cdot 2^2\zeta(-2)=1^2-2^2+3^2-4^2+\dots=0.$$

Thus,

$$\zeta(-2)-2\cdot 2^2\zeta(-2)=(1-2^3)\zeta(-2)=-7\zeta(-2)=0 \implies \zeta(-2)=0,$$

as expected.