A Pointed Assignment

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1. $P(k)_n$ is the number of maps from the k set (i.e. "the" set with k elements) to "the" n set (set with n elements) multiplied by the number of total orderings on n, since we choose both such a map and a total ordering on n. The number of maps $f : k \to n$ is just n^k since each element of k has n possible images, and these are all chosen independently (i.e. we allow repetition). The number of total orderings is of course $n!$. So the product of these gives $|P(k)_n| = n^k \cdot n!$.

2. We have in general $|P(k)|(z) = \sum_{n\geq 0} \frac{|P(k)_n|z^n}{n!}$. In this case, the $n!$ factors cancel (that is, the total ordering on $n!$ makes this a case we could think of as an ordinary generating series), so we have:
$$|P(k)|(z) = \sum_{n\geq 0} \frac{n!\cdot n^k z^n}{n!} = \sum_{n\geq 0} n^k z^n$$

3. In the special case where $k = 0$ the generating function above is:
$$|P(0)|(z) = \sum_{n\geq 0} n^0 z^n = \sum_{n\geq 0} z^n = \frac{1}{1-z}$$

4. To put an $N\Psi$-structure on a set S is to put an $A^*A\Psi$-structure on it, and this means, by the definition of the A^* operator, to choose an element x of S and then put an $A\Psi$-structure on $S\setminus\{x\}$. Now, to put an $A\Psi$-structure on $S\setminus\{x\}$ is, by the definition of the A operator, to put a Ψ-structure on $(S\setminus\{x\})+1$, that is, $S\setminus\{x\}$ with a single point added to it. So, to put an $N\Psi$-structure on S is to choose a point $x \in S$, remove it from S, then add a new point to the resulting set, and put a Ψ-structure on the set thus created. This is equivalent to identifying a special point of S (since we have a natural isomorphism beween S and the resulting set which sends every element of $S\setminus x$ to itself, and x to the new one-point set denoted 1) and then putting a Ψ-structure on it.

5. To put an $NP(k)$-structure on a set S is to specify a point $x \in S$ and also put a $P(k)$-structure on it. Now, a $P(k)$-structure on S is a k-pointing - that is, a labelling of k points (possibly with repetition) of S by numbers $1 \ldots k$. On the other hand, a $P(k+1)$-structure is a labelling of $k+1$ elements of S by numbers $1 \ldots k+1$. There is a natural way to define an isomorphism between such structures. Given an $NP(k)$-structure on S, construct a $P(k+1)$-structure on S by assigning the numbers $1 \ldots k$ to the same points in the $(k+1)$-pointing as in the k-pointing, and assigning the number $k+1$ to the specially identified point from the $NP(k)$-structure. This is clearly reversible, hence an isomorphism. In particular, it is natural since there is a unique natural choice for which element of $k+1$ to assign to the special point. Thus, by thinking of one assignment of labels from $k+1$ as an assignment of labels in k to a pointed set, we have
$$NP(k) \cong P(k+1)$$

6. We have seen previously that the effect of the A and A^* operators on the generating functions corresponding to a structure type Ψ is, respectively, $|A\Psi|(z) = \frac{\mathrm{d}}{\mathrm{d}z}|\Psi|(z)$ and $|A^*\Psi|(z) = z|\Psi|(z)$. So combining these, and the existence of the isomorphism above gives that:

$$|P(k+1)|(z) = |NP(k)|(z) = |A^*AP(k)|(z) = z\frac{\mathrm{d}}{\mathrm{d}z}|P(k)|(z)$$

7. By part 6, we have that $|P(1)|(z) = z\frac{\mathrm{d}}{\mathrm{d}z}|P(0)|(z)$, but since by part 3 we know that $|P(0)|(z) = \frac{1}{1-z}$, we find that:

$$|P(1)|(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{1-z}\right) = z\left(-\frac{-1}{(1-z)^2}\right) = \frac{z}{(1-z)^2}$$

8. Now we come to the point of this assignment - evaluating divergent sums using our generating function, and using Abel sums. The first says that if we have the expression above equal to —$P(1)|(z)$, then $|P(1)|(-1) = \frac{-1}{(1-(-1))^2} = -\frac{1}{2^2} = -\frac{1}{4}$. On the other hand, we know from part 2 that

$$|P(1)|(z) = \textstyle\sum_{n\geq 0}(n)^1 z^n$$

But then, if $z = -1$, this gives that

$$|P(1)|(-1) = \textstyle\sum_{n\geq 0} n\cdot(-1)^n = -1+2-3+\dots$$

Using these two expressions, we could claim that $-(-1+2-3+\dots) = -\frac{1}{4}$. This is the same as what we want, namely that $1-2+3-4+\dots = \frac{1}{4}$.

9. We observe that the sum above does not actually converge, since the point $z = -1$ is not strictly inside the radius of convergence for the function $|P(1)|(z)$ written as a power series expanded about $z = 0$. This is because the function has a pole at $z = 1$, so the radius of convergence is 1, but this function is analytic everywhere else in the complex plane. So the function can be analytically continued to $z = -1$, though the power series does not converge there. This is exactly what the Abel sum:

$$A\textstyle\sum_{n=1}^{\infty}(-1)^{n+1}n = -\lim_{t\nearrow 1}\sum_{n=1}^{\infty} t^n(-1)^n\cdot n$$

is doing: this is an analytic continuation of $|P(1)|(z)$ to $z = -1$ along the negative real axis. This is

$$-\lim_{t\nearrow 1}\textstyle\sum_{n=1}^{\infty}(-t)^n\cdot n = -\lim_{t\nearrow 1}|P(1)|(-t) = -\lim_{t\nearrow 1}\frac{-t}{(1+t)^2} = \frac{1}{4}$$

So in fact the Abel sum of the series in question is indeed the value we found using $|P(1)|(z)$.

10. We have that $|P(2)|(z) = z\frac{\mathrm{d}}{\mathrm{d}z}|P(1)|(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{(1-z)^2}\right)$, using parts 6 and 7 respectively. This means that

$$|P(2)|(z) = \frac{(1-2z+z^2)+(2z-2z^2)}{(1-z)^4} = \frac{1-z^2}{(1-z)^4} = \frac{1+z}{(1-z)^3}$$

But on the other hand, we know by part 2 that

$$|P(2)|(z) = \sum_{n\geq 0} n^2 z^n$$

If we evaluate this sum at $z = -1$, we get the alternating sum $-1^2 + 2^2 - 3^2 + 4^2 \ldots$, so the Abel sum of the series $1^2 - 2^2 + 3^2 - 4^2 + \ldots$ will be the negative of $|P(2)|(-1)$, by the same reasoning as above, namely that $|P(2)|(z)$ as given above is an analytic function on all of $\mathbb{C}$ except for a pole of order 3 at $z = 1$. Thus we can extend analytically in a unique way to $z = -1$, and so:

$$\begin{aligned} A\sum_{n=1}^{\infty}(-1)^{n+1} \cdot n^2 &= -\lim_{t\nearrow 1} t^n(-1)^n \cdot n^2 \\ &= -\lim_{t\nearrow 1}(-t)^n \cdot n^2 \\ &= -\lim_{t\nearrow 1} |P(2)|(-t) \\ &= -\lim_{t\nearrow 1} \frac{(1-t)}{(1+t)^3} \\ &= 0 \end{aligned}$$

Now using Euler's approach, we would say that $\zeta(-2) = 1^2 + 2^2 + 3^2 + \ldots$ and we can also get that $4\zeta(-2) = 2^2 + 4^2 + 6^2 + \ldots$, since each term here is 4 times the corresponding term in $\zeta(-2)$. Thus, if we subtract twice this second series, we should get the alternating series from above: $(1 - 2(4\zeta(-2)) = 1^2 - 2^2 + 3^2 - 4^2 + \ldots$ or in other words

$$-7(1^2 + 2^2 + 3^2 + \ldots) = 1^2 - 2^2 + 3^2 - 4^2 \ldots = 0$$

In other words, $\zeta(-2) = 1^2 + 2^2 + 3^2 + \ldots = 0$.