$1 + 2 + 3 + \cdots = -\frac{1}{12}$

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1. *Totally ordered, k-pointed sets.*

There are $n!$ ways to totally order the set n, and n^k functions $f\colon k \to n$, so

$$|P(k)_n| = n!n^k.$$

2. *Generating function*

$$|P(k)| = \sum_{n\geq 0} \frac{|P(k)_n|}{n!} z^n = \sum_{n\geq 0} n^k z^n.$$

3. *Totally ordered sets.*

Since 0-pointing amounts to doing nothing (in one way), $P(0)$ is isomorphic to the structure type "being a totally ordered finite set", and that has generating function

$$|P(0)|(z) = \sum_{n\geq 0} z^n = \frac{1}{1-z}.$$

4,5. *$N\Phi$-structures.*

To put a $N\Phi$-structure on a set S means to choose $s \in S$ and to put an $A\Phi$-structure on $S - \{s\}$. This means to choose $s \in S$ and to put a Φ structure on $(S - \{s\}) + 1$ or, equivalently, to choose $s \in S$ and to put a Φ-structure on S. This is equivalent to pointing S and putting a Φ-structure on it. Hence,

an $N\Phi$-structure is a pointed Φ-structure.

6,7. *k-pointed Φ-structures.*

It is obvious that an $N^k\Phi$-structure is isomorphic to a k-pointed Φ-structure, so a $P(k)$ structure isomorphic to an $N^kP(0)$-structure, and so $NP(k) \simeq P(k+1)$.

The decategorification of this isomorphism is

$$z\frac{d}{dz}|P(k)|(z) = |P(k+1)|(z),$$

so

$$|P(1)|(z) = z\frac{d}{dz}|P(0)|(z) = \frac{z}{(1-z)^2}.$$

8,9,10. *Euler's trick.*

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots = |P(1)|(-1) = -\frac{1}{4}.$$

We can prove this rigorously using Abel sums like this:

$$A\sum_{n\geq 0}(-1)^{n+1}n = -\lim_{t\uparrow 1}\sum_{n\geq 0}(-t)^n n = -\lim_{t\downarrow -1}\sum_{n\geq 0} t^n n = -\lim_{t\downarrow -1}\sum_{n\geq 0} t\frac{d}{dt}t^n =$$
$$= -\lim_{t\downarrow -1} t\frac{d}{dt}\sum_{n\geq 0} t^n = -\lim_{t\downarrow -1} t\frac{d}{dt}\frac{1}{1-t} = \lim_{-t\downarrow -1}\frac{t}{(1-t)^2} = \frac{1}{4}.$$

Finally, since

$$|P(2)|(z) = z\frac{d}{dz}|P(1)|(z) = z\Big[\frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3}\Big] = \frac{z(1+z)}{(1-z)^3}$$

we have

$$(1-2^3)\zeta(-2) = |P(2)|(-1) = 0,$$

so $\zeta(-2) = 0$.