Now let's prove this:

**Theorem** - Suppose \( G \) is a group & \( M \) is a connected triangulated manifold with chosen vertex \( * \). Let \( \pi_1(M) \) be the fundamental group of \( M \) & let \( \mathcal{A}_0(M) \) be the set of flat \( G \)-connections on \( M \) & let \( \mathcal{G}_0(M) \) be the subgroup of the group \( \mathcal{G}(M) \) of gauge transformations consisting of gauge transformations \( g \) with \( g(*) = 1 \). Then:

\[
\mathcal{A}_0(M)/\mathcal{G}_0(M) \cong \text{hom}(\pi_1(M), G)
\]

**Proof** - Recall that for any triangulated manifold \( M \) there's the path groupoid \( \mathcal{P}M \) with

- **Objects** = vertices of \( M \)
- **Morphisms** = edge paths in \( M \)  
  \((ee^{-1} = 1, e^{-1}e = 1)\)

If \( M \) is connected, there's just one isomorphism class of objects in \( \mathcal{P}M \), so we can get an equivalent groupoid (a skeleton of \( \mathcal{P}M \)) by taking one object \( * \in \mathcal{P}M \) & forming the loop group \( \mathcal{L}M \) with

- **Object** = *
- **Morphisms** = edge paths \( \gamma : * \to * \)
  i.e. "edge loops based at *"

We have an equivalence of categories

\[
\mathcal{L}M \leftrightarrow \mathcal{P}M
\]
Next, recall that for any triangulated manifold $M$ there's a fundamental groupoid $\Pi_1 M$ with

- **objects** = vertices of $M$
- **morphisms** = simplicial homotopy classes of edge paths in $M$

Again, when $M$ is connected we can pick any object $* \in \Pi_1 M$ and define a skeleton of $\Pi_1 M$, the fundamental group $\pi_1 M$ with

- **object** = $*$
- **morphisms** = simplicial homotopy classes of loops $\gamma : * \to *$.

We have an equivalence:

$$\Pi_1 M \leftrightarrow \Pi_1 M$$

Note we have a commutative square:

$$\begin{array}{ccc}
\Omega M & \leftrightarrow & PM \\
\downarrow & & \downarrow \\
\Pi_1 M & \leftrightarrow & \Pi_1 M
\end{array}$$

where the downward arrows are quotient maps (essentially surjective and full). This square gives another square

$$\begin{array}{ccc}
\text{hom}(\Omega M, G) & \leftrightarrow & \text{hom}(PM, G) \\
\uparrow & & \uparrow \\
\text{hom}(\Pi_1 M, G) & \leftrightarrow & \text{hom}(\Pi_1 M, G)
\end{array}$$
Note: if \( X \) & \( Y \) are categories, \( \text{hom}(X,Y) \) is a category.

where

objects = functors \( f: X \to Y \)  

morphisms = natural transformations  

(Cet is a closed category)

If \( X \) & \( Y \) are groupoids this \( \text{hom}(X,Y) \) is a groupoid.

Note:

\[
\text{hom}(PM,G) \text{ has as } \begin{cases} 
\text{objects: G-connections on } M : A \in \mathcal{A}(M) \\
\text{morphisms: } \text{gauge transformations} \\
g: A \to A' \text{ with } g \in G(M) 
\end{cases}
\]

\[
\text{hom}(\Pi M, G) \text{ has as } \begin{cases} 
\text{objects: flat G-connections on } M : A \in \mathcal{A}_0(M) \\
\text{morphisms: } \text{gauge transformations} \\
g: A \to A' \text{ with } g \in G(M) 
\end{cases}
\]

So

\[
\mathcal{A}_0(M)/G(M) = \text{isomorphism classes of objects in } \text{hom}(\Pi M, G)
\]

and

\[
\mathcal{A}_0(M)/G_0(M) = \text{objects in } \text{hom}(\Pi M, G) \text{ mod isomorphisms } g \text{ with } g(*) = 1
\]

Similarly:

\[
\text{hom}(\Pi M, G) \text{ has as } \begin{cases} 
\text{objects: } \text{homomorphisms } f: \Pi M \to G \\
\text{morphisms: } \text{elts } g \in G \text{ with } f'(x) = g f(x) g^{-1}
\end{cases}
\]

\[
\xrightarrow{\text{i.o.}} (f \circ g) & \xrightarrow{g} f \circ g & \xrightarrow{g} f(x) \xrightarrow{\text{for } x \in \Pi M}
\]
To show that $\mathbb{A}_0(M)/G_0(M)$ is $\cong$ the set of homomorphisms $f: \Pi_1 M \to G$, note: \{objects in $\text{hom}(\Pi_1 M, G)$ mod isos $g$ with $g(*) = 1 \}_{/G} \cong \{ \text{homomorphisms } f: \Pi_1 M \to G \}_{/G}$ while \{objects in $\text{hom}(\Pi_1 M, G)$ mod isos $g$ w. $g(*) = 1 \}_{/G} \cong \mathbb{A}_0(M)/G_0(M)$.

These sets are isomorphic because $\Pi_1 M$ & $\Pi_1 M$ are equivalent (and we are modding out by the same natural isos. in both $\text{hom}(\Pi_1 M, G)$ and $\text{hom}(\Pi_1 M, G)$).

We've shown much more, e.g.: for any triangulated manifold $M$,

\[ \mathbb{A}(M)/G(M) = \{ \text{nat. iso classes of functors } A: PM \to G \}_{/G} \]
\[ \mathbb{A}_0(M)/G(M) = \{ \text{nat. iso classes of functors } A: \Pi_1 M \to G \}_{/G} \]

and if $M$ is connected, then

\[ \mathbb{A}(M)/G(M) = \{ \text{conjugacy classes of homomorphisms } f: \Omega M \to G \}_{/G} \]

(where $f, f': \Omega M \to G$ are conjugate if $f'(x) = g f(x) g^{-1}$, $\exists g \in G$).

\[ \mathbb{A}_0(M)/G(M) = \{ \text{conjugacy classes of homomorphisms } f: \Pi_1 M \to G \}_{/G} \]

Vast amounts of energy have been spent studying this last space for various choices of $M$ & $G$ - it's called the "moduli space of flat connections."
Recap:

For any finite group $G$ we get a 2D TQFT $Z$ from the group algebra $\mathbb{C}[G]$, and if $M$ is a compact oriented connected 2-manifold,

$$Z(M) = \sum_{\mathcal{A}_0(M)/\mathcal{G}_0(M)} \frac{1}{|G|^{2g-1}}$$

where $g$ is the genus of $M$, i.e. it's a “path integral” over the space $\mathcal{A}_0(M)/\mathcal{G}_0(M)$ of flat connections mod gauge transformations that equal 1 at some chosen $* \in M$. The genus $g$ is related to the Euler characteristic $\chi(M)$ by

$$\chi(M) = 2 - 2g$$

so

$$2g = 2 - \chi(M)$$

$$2g - 1 = 1 - \chi(M)$$

and

$$Z(M) = \sum_{\mathcal{A}_0(M)/\mathcal{G}_0(M)} |G|^{\chi(M) - 1}$$

The annoying $|G|^{\chi(M) - 1}$ comes from $\mathcal{G}(M)/\mathcal{G}_0(M) \cong G$ & indeed we would have

$$Z(M) = \sum_{\mathcal{A}_0(M)/\mathcal{G}(M)}$$

But this isn't true even though $\mathcal{A}_0(M)/\mathcal{G}(M) \cong \frac{\mathcal{A}_0(M)/\mathcal{G}_0(M)}{G}$

since it's not true that

\[ |A_0(M)/G(M)| = \frac{|A_0(M)/G_0(M)|}{|G|} \]

unless \( G \) acts freely on \( A_0(M)/G(M) \).

(suggests we really should be using groupoid cardinality and the weak quotient)

If \( G \) acts freely on \( A_0(M)/G_0(M) \), then the formula holds and

\[ Z(M) = |A_0(M)/G(M)| \cdot |G| \cdot \chi(M) \]

In general, we only have

\[ Z(M) = \frac{|A_0(M)/G_0(M)|}{|G|} \]

Or (see last year's notes) we could form the weak quotient \( A_0(M)/G(M) \) - a groupoid, and use groupoid cardinality to get:

\[ Z(M) = |A_0(M)/G(M)| \cdot |G| \cdot \chi(M) \]

\( A_0(M)/G(M) \) is called the moduli stack of flat connections, and keeps track of how certain flat connections get mapped to themselves by gauge transformations at the special vertex \( * \). These are called reducible connections.
$|G|^\chi(M)$ resembles $e^{-S}$ & indeed the action in 2d gravity is (proportional to) $\chi(M)$. In any dimension, there's an "Euler TQFT" with $Z(M) = \alpha^\chi(M)$ for any closed manifold $M$ (and any fixed $\alpha \in \mathbb{C}$).

All this stuff works in any dimension. In dimension 3, we get a TQFT $Z$ from the group 2-algebra $\text{Vect}[G]$ which has a basis of objects $C_g$ ($g \in G$) with

$$C_g \otimes C_h = C_{gh}$$

& this was studied by Dijkgraaf & Witten (the "Dijkgraaf-Witten model") & later by Dan Freed & Frank Quinn. They saw if $M$ is a compact oriented 3-manifold,

$$Z(M) = |A_0(M)\otimes G(M)|$$

What happened to the $|G|^\chi(M)$? Answer: $\chi(M) = 0$ for every compact oriented 3-manifold. In any dimension $n$

$$\chi(M) = \sum_{i=1}^{n} (-1)^i \dim |H_i(M, \mathbb{R})|$$

& $H_i(M, \mathbb{R}) \cong H^*_{n-i}(M, \mathbb{R})$ (Poincaré duality)

so we get 0 when $n$ is odd.
\[ \dim H_0 - \dim H_1 + \dim H_2 - \dim H_3 \]

So in fact \( (G)^{\chi(m)} \) is really there; it's just hiding.

In fact, in any dimension we get a TQFT with

\[ Z(M) = |\mathcal{A}_0(M) / \mathcal{G}(M)| \ |G|^{\chi(M)} \]

but people haven't constructed them using \( 2\text{Vect}[G] \), \( 3\text{Vect}[G] \), \( 4\text{Vect}[G] \), ... etc., except in dimensions \( \leq 4 \).

(Marco Mackaay has built the 4d TQFT using \( 2\text{Vect}[G] \).)

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**Twisting:**

Dijkgraaf & Witten showed how to modify or "twist" the TQFTs we've been discussing, using group cohomology.

For example, in 2 dimensions, let's take \( C[G] \) with product:

\[ \delta_g \delta_h = \delta_{gh} \]

where

\[ \delta_g (h) = \begin{cases} 0 & g \neq h \\ 1 & g = h \end{cases} \]

and then define a new "twisted" product, \( * \), as follows...
\[ \delta_g \ast \delta_h = c(g,h) \delta_{gh} \]

where
\[ c : G^2 \longrightarrow C - \{ 0 \} . \]

To get a TQFT we want this to yield a new semisimple algebra - in particular, it had better be associative.
Let's see when this is true:

\[
(\delta_g \ast \delta_h) \ast \delta_k = \delta_g \ast (\delta_h \ast \delta_k)
\]

\[
\begin{align*}
&\quad c(g,h) \delta_{gh} \ast \delta_k \\
&\quad = c(h,k) \delta_g \ast \delta_{hk} \\
&\quad = c(g,h) c(gh,k) \delta_{ghk} \\
&\quad = c(h,k) c(g,hk) \delta_{ghk}
\end{align*}
\]

so we need \( c \) to be a \underline{2-cocycle}, i.e.

\[ c(g,h) c(gh,k) = c(h,k) c(g,hk) \]

or better:

\[ c(h,k) c(gh,k)^{-1} c(g,hk) c(g,h)^{-1} = 1 \]

\underline{omit first one, multiply first pair, multiply second pair, omit last one.}

To keep \( \delta_1 \) being the multiplicative identity, we also need

\[ \delta_1 \ast \delta_g = c(1,g) \delta_g = \delta_g \]
\[ \delta_g \ast \delta_1 = c(g,1) \delta_g = \delta_g \]
i.e.
\[ c(1,g) = c(g,1) = 1 \]

i.e. \( c \) is a \underline{normalized cocycle}.

Next time, we'll twist the \underline{associator in Vect[G]} using a \underline{normalized 3-cocycle}.