Quantum Gravity Seminar
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Checking that Cat is a 2-Category

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Check that Cat is a (strict) 2-category.

What's in a 2-category?

A 2-category has objects, morphisms, and 2-morphisms. In Cat, these will be given as follows:

- *Objects.* Categories will be the bjects.
- *Morphisms.* Functors between categories will be the morphisms.
- *2-Morphisms.* Natural transformations will be the 2-morphisms.

Is Cat even a category?

We check the basic requirements of a category:

- *Identity morphisms.* For any category C in Cat, we get an identity functor $1_C\colon C \to C$. This is the functor which does nothing to the objects and morphisms of C.
- *Composition.* For any pair of functors $F\colon C \to D$ and $G\colon D \to E$, we define $FG\colon C \to E$ to be the composite of F and G. We just need to make sure that FG is a functor: that is, it preserves identities and composition.
  - *Identities.* For any object x in C,
  $$FG(1_x) \;=\; G(1_{F(x)}) \;=\; 1_{FG(x)}.$$
  - *Composition.* For any pair of morphisms $f\colon x \to y$ and $g\colon y \to z$ in C,
  $$FG(fg) \;=\; G(F(f)F(g)) \;=\; G(F(f))G(F(g)) \;=\; FG(f)FG(g).$$

Now we need the left and right unit laws to hold, along with the associative law.

- *Left and right unit laws.* For any functor $F\colon C \to D$, it is clear that $1_C\mathrm{F} = \mathrm{F} = \mathrm{F}1_D$.
- *Associative law.* For any triple of functors $F\colon A \to B$, $G\colon B \to C$, $H\colon C \to D$, $F(GH) = (FG)H$. This follows from the associativity of the functions on objects and morphisms which comprise our functors.

So Cat is a category! A good start.

Defining the compositions.

Now we need to pay attention to the 2-morphisms. In this case, the natural transformations. We can compose our 2-morphisms in two ways, vertically and horizontally. We want to show that these composites are natural transformations and that the interchange law holds between these two compositions.

- *Vertical composition.* Given functors $F, G, H\colon C \to D$ and natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon G \Longrightarrow H$, we want to define $\alpha\beta\colon F \Longrightarrow H$.

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"{name=U}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, "h"'{name=D}] & B \arrow[from=U, to=M, phantom, "\Downarrow\alpha"] \arrow[from=M, to=D, phantom, "\Downarrow\beta"] \end{tikzcd}$$

Since α and β are natural transformations, we have for every x in C, functions

$$\alpha_x\colon F(x) \to G(x) \textit{ and } \beta_x\colon G(x) \to H(x)$$

such that for all morphisms $f\colon x \to y$ in C, the following diagrams commute:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"'] & F(y) \arrow[d, "\alpha_y"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd} \qquad \begin{tikzcd} G(x) \arrow[r, "G(f)"] \arrow[d, "\beta_x"'] & G(y) \arrow[d, "\beta_y"] \\ H(x) \arrow[r, "H(f)"] & H(y) \end{tikzcd}$$

Given that the diagrams above commute and defining the composite $(\alpha\beta)_x\colon F(x) \to H(x)$ for any x in C as

$$F(x) \xrightarrow{\alpha_x} G(x) \xrightarrow{\beta_x} H(x)$$

it is clear that the following diagram commutes:

$$
\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"'] & F(y) \arrow[d, "\alpha_y"] \\
G(x) \arrow[r, "G(f)"] \arrow[d, "\beta_x"'] & G(y) \arrow[d, "\beta_y"] \\
H(x) \arrow[r, "H(f)"] & H(y)
\end{tikzcd}
$$

Thus, the vertical composite is natural.

- *Horizontal composition.* Given functors $F, F' \colon C \to D$, $G, G' \colon D \to E$, and natural transformations $\alpha \colon F \implies F'$ and $\beta \colon G \implies G'$, we want to define $\alpha \circ \beta \colon F \circ G \implies F' \circ G'$. Given any object x in C, we have

  $$\alpha_x \colon F(x) \to F'(x).$$

  Under the functors G and G', this gets mapped to:

  $$G(F(x)) \xrightarrow{G(\alpha_x)} G(F'(x))$$

  and

  $$G'(F(x)) \xrightarrow{G'(\alpha_x)} G'(F'(x)).$$

  By the naturality of β, we obtain another commuting diagram which allows us to make a well-defined choice for our composite:

  $$
  \begin{tikzcd}
  G(F(x)) \arrow[r, "G(\alpha_x)"] \arrow[d, "\beta_{F(x)}"'] & G(F'(x)) \arrow[d, "\beta_{F'(x)}"] \\
  G'(F(x)) \arrow[r, "G'(\alpha_x)"] & G'(F'(x)).
  \end{tikzcd}
  $$

  Now we just need to check that our horizontal composition is natural. So we need the following diagram to commute:

  $$
  \begin{tikzcd}
  G(F(x)) \arrow[r, "G(F(f))"] \arrow[d, "\alpha\circ\beta_x"'] & G(F(y)) \arrow[d, "\alpha\circ\beta_y"] \\
  G'(F'(x)) \arrow[r, "G'(F'(f))"] & G'(F'(y)).
  \end{tikzcd}
  $$

We can extend this diagram, using the definition of our horizontal composite, to the following diagram:

$$
\begin{tikzcd}
G(F(x)) \arrow[r, "G(F(f))"] \arrow[d, "G(\alpha_x)"'] & G(F(y)) \arrow[d, "G(\alpha_y)"] \\
G(F'(x)) \arrow[r, "G(F'(f))"] \arrow[d, "\beta_{F'(x)}"'] & G(F'(y)) \arrow[d, "\beta_{F'(y)}"] \\
G'(F'(x)) \arrow[r, "G'(F'(f))"] & G'(F'(y)).
\end{tikzcd}
$$

The upper square commutes since we have

$$F(f)\alpha_y \;=\; \alpha_x F'(F)$$

by the naturality of α, and we can then apply the functor G to this equation. The lower square commutes by the naturality of β. Thus the horizontal composite is natural.

The interchange law.

Given objects C, D, E in Cat and morphisms F, $F', F'': C \longrightarrow D$ and G, $G', G'': D \longrightarrow E$, if we have 2-morphisms $\alpha: F \Longrightarrow F'$, $\alpha': F' \Longrightarrow F''$, $\beta: G \Longrightarrow G'$, and $\beta': G' \Longrightarrow G''$, then we want the following composition diagram to be unambiguous:

$$
\begin{tikzcd}
C \arrow[r, bend left=60, "F"] \arrow[r] \arrow[r, bend right=60, "F"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha'"] & D \arrow[r, bend left=60, "G"] \arrow[r] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & E
\end{tikzcd}
$$

By the naturality of vertical composition we get the following commutative rectangle:

$$
\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\alpha_x"'] & Fy \arrow[d, "\alpha_y"] \\
F'x \arrow[r, "F'f"] \arrow[d, "\alpha'_x"'] & F'y \arrow[d, "\alpha'_y"] \\
F''x \arrow[r, "F''f"] & F''y
\end{tikzcd}
$$

If we apply our 3 functors from D to E to this rectangle, we get a big commuting cube, which tells us that the interchange law

$$\alpha\alpha' \circ \beta\beta' \;=\; (\alpha \circ \beta)(\alpha' \circ \beta')$$

holds. So we inspect this cube and see that it commutes by the naturality of vertical composition, horizontal composition and functoriality of G, G' and G''.

$$\begin{tikzcd}
 & & G''Fx \arrow[r, "G''Ff"] \arrow[d, "G''\alpha_x"'] & G''Fy \arrow[d, "G''\alpha_y"] \\
 & & G''F'x \arrow[r, "G''F'f"] \arrow[d, "G''\alpha'_x"'] & G''F'y \arrow[d, "G''\alpha'_y"] \\
 & & G''F''x \arrow[r, "G''F''f"] & G''F''y. \\
 & G'Fx \arrow[r, "G'Ff"] \arrow[d, "G'\alpha_x"'] \arrow[uuur, "\beta'_{Fx}"] & G'Fy \arrow[d, "G'\alpha_y"] \arrow[uuur, "\beta'_{Fy}"] & \\
 & G'F'x \arrow[r, "G'F'f"] \arrow[d, "G'\alpha'_x"'] \arrow[uuur, "\beta'_{F'x}"] & G'F'y \arrow[d, "G'\alpha'_y"] \arrow[uuur, "\beta'_{F'y}"] & \\
 & G'F''x \arrow[r, "G'F''f"] \arrow[uuur, "\beta'_{F''x}"] & G'F''y. \arrow[uuur, "\beta'_{F''y}"] & \\
GFx \arrow[r, "GFf"] \arrow[d, "G\alpha_x"'] \arrow[uuur, "\beta_{Fx}"] & GFy \arrow[d, "G\alpha_y"] \arrow[uuur, "\beta_{Fy}"] & & \\
GF'x \arrow[r, "GF'f"] \arrow[d, "G\alpha'_x"'] \arrow[uuur, "\beta_{F'x}"] & GF'y \arrow[d, "G\alpha'_y"] \arrow[uuur, "\beta_{F'y}"] & & \\
GF''x \arrow[r, "GF''f"] \arrow[uuur, "\beta_{F''x}"] & GF''y. \arrow[uuur, "\beta_{F''y}"] & &
\end{tikzcd}$$

After staring at this for a while we can believe that the interchange law holds.

Is Cat a 2-category?

The category $\mathrm{Hom}(C, D)$

The first thing we need to show is that given two objects C, D in Cat, we get a category $\mathrm{Hom}(C, D)$. The objects of $\mathrm{Hom}(C, D)$ are functors and the morphisms are natural transformations. Luckily, we already know how to compose our morphisms. For each triple F, G, H of objects in $\mathrm{Hom}(C, D)$ we have a function

$$\circ\colon \mathrm{Hom}(F, G) \times \mathrm{Hom}(G, H) \Longrightarrow \mathrm{Hom}(F, H)$$

which is just our vertical composition of natural transformations that we defined above. Also, for any object F in $\mathrm{Hom}(C, D)$, we have an identity natural transformation

$$\mathbf{1}_F\colon F \Longrightarrow F.$$

Given any natural transformation $\alpha\colon F \Longrightarrow G$, we have

$$\mathbf{1}_F\alpha \;=\; \alpha \;=\; \alpha\mathbf{1}_G.$$

This is clear since for any object x in C, the natural transformations $\mathbf{1}_F$ andα give us morphisms

$$\mathbf{1}_{F_x}\colon Fx \Longrightarrow Fx \;\; \textit{and} \;\; \alpha_x\colon Fx \Longrightarrow Gx.$$

We know that morphisms compose and $\mathbf{1}_{F_x}$, the component of our identity natural transformation will behave as an identity.

We need to show that our morphisms, the natural transformations, are associative. Given any four objects F, G, H, I in $\mathrm{Hom}(C, D)$ and natural transformations $\alpha\colon F \Longrightarrow G, \beta\colon G \Longrightarrow H, \gamma\colon H \Longrightarrow I$, we know that

$$(\alpha\beta)\gamma \;=\; \alpha(\beta\gamma),$$

since the components of natural transformations are morphisms, which are associative.
So, we see that $\mathrm{Hom}(C, D)$ is indeed a category.

The composition functor

Given any three objects C, D, E in Cat, we define a composition functor

$$\circ\colon \mathrm{Hom}(C, D) \times \mathrm{Hom}(D, E) \Longrightarrow \mathrm{Hom}(C, E),$$

by

$$(F, G) \;\longmapsto FG$$

- regular functor composition - and

$$(\alpha, \beta) \;\longmapsto \alpha \circ \beta$$

- horizontal composition. While our composition notation is becoming heavily overloaded, we will only really concern ourselves with the vertical and horizontal compostion symbols defined in the second section of this note, where vertical composition is juxtaposition and horizontal composition is $\circ$.

Now we want to check that this 'composition functor' is actually a functor. Given any object (F, G) in $\mathrm{Hom}(C, D) \times \mathrm{Hom}(D, E)$, we have

$$\circ(\mathbf{1}_F, \mathbf{1}_G) \;=\; \mathbf{1}_F \circ \mathbf{1}_G,$$

the identity on FG.
We also need to check that the functor respects composition. But this is just the interchange law proved earlier! So we have a functor.

Almost there!

Since Cat is a strict category, our associator natural ismorphisms will all be just the identity natural isomorphism since the associative law for functors is an equation as we saw earlier. A similar statement is true for the left and right unit natural isomorphisms.

So, Cat is a 2-category.