Cat is a 2-Category

Jeffrey Morton

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Theorem 1 *The collection* Cat *of all categories, together with functors and natural transformations, is a (strict) 2-category.*

Proof: *We first define the 2-category and then show it satisfies the usual axioms. We define it to contain:*

- ***Objects****: Categories*
- ***Morphisms****: Functors*
- ***2-Morphisms****: Natural transformations*

To make **Cat** *into a 2-category, we need to show that left and right unitors, and an associator. However, this is trivial: in fact, it suffices to take all of these to be the identity, making* **Cat** *a strict 2-category. To verify that this choice works, we check that: for any categories C_1 and C_2, $Hom(C_1, C_2)$ is a category; that composition is a functor between such hom-categories; and that the unitors and associator satisfy the requisite identities. First is the "triangle identity", namely the condition that the diagram*

$$\begin{tikzcd}
(G \circ 1) \circ F \arrow[r, "a_{G,1,F}"] \arrow[d, "l \otimes 1_F"'] & G \circ (1 \circ F) \arrow[dl, "1_G \otimes r"] \\
G \circ F &
\end{tikzcd} \tag{1}$$

commute. This holds since all three maps a, r, and l, are just identity maps. The associator satisfies the pentagon identity:

$$\begin{tikzcd}
 & (F \circ G) \circ (H \circ J) \arrow[dr, "a_{F,G,H \circ J}"] & \\
((F \circ G) \circ H) \circ J \arrow[ur, "a_{F \circ G,H,J}"] \arrow[d, "a_{F,G,H} \circ 1_J"'] & & F \circ (G \circ (H \circ J)) \\
(F \circ (G \circ H)) \circ J \arrow[r, "a_{F,G \circ H,J}"'] & F \circ ((G \circ H) \circ J) \arrow[ur, "1_F \circ a_{G,H,J}"'] &
\end{tikzcd} \tag{2}$$

for the same reason.

Next we show that for any categories C_1 and C_2, hom(C_1, C_2) is a category. To do this, we need to check that the collection of natural transformations is closed under ("vertical") composition, and that there is an identity transformation for each functor. The vertical composition of two natural transformations works as follows: each natural transformation $\alpha : F \to G$ gives, for each object $x \in C_1$, a morphism $\alpha(x) \in C_2$, so that for every morphism $f : x \to y$ in C_1, the diagram

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha(x)"'] & F(y) \arrow[d, "\alpha(y)"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd} \tag{3}$$

commutes. But then, vertical composition of natural transformations $\alpha : F \to G$ and $\beta : G \to H$ gives a diagram like this for each morphism f:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha(x)"] & F(y) \arrow[d, "\alpha(y)"] \\ G(x) \arrow[r, "G(f)"] \arrow[d, "\beta(x)"] & G(y) \arrow[d, "\beta(y)"] \\ H(x) \arrow[r, "H(f)"] & H(y) \end{tikzcd} \tag{4}$$

and since each square commutes, so does the whole diagram, so that $\beta \circ \alpha : F \to H$ is a natural transformation. This composition is associative since the composition $\beta(x) \circ \alpha(x)$ is associative for each x. Likewise, there is a unit natural transformation $1_F : F \to F$ for which $1(x) = 1_x$ for every x.

Next we need to see that ("horizontal") composition of functors $\circ : hom(A, B) \times hom(B, C) \to \mathrm{hom}(A, C)$ is itself functorial, we need to check that the composite of two functors is indeed a functor; that the operation $\circ$ preserves source, target, and ("vertical") composition of natural transformations; and that the associativity and unit axioms hold.

Now, since functor $F : C_1 \to C_2$ consists of a set map $\mathrm{ob}(F) : \mathrm{ob}(C_1) \to \mathrm{ob}(C_2)$ between the sets of objects together with a set map $\mathrm{mor}(F) : \mathrm{mor}(C_1) \to \mathrm{mor}(C_2)$ between the morphisms, such that source and target maps s and t, and composition, are preserved. Given two functors $F : C_1 \to C_2$ and $G : C_2 \to C_3$, we then can compose the set maps for objects and morphisms separately, and define $\mathrm{ob}(G \circ F) = \mathrm{ob}(G) \circ \mathrm{ob}(F)$ and $\mathrm{mor}(G \circ F) = \mathrm{mor}(G) \circ \mathrm{mor}(F)$.

To see this is a functor, note that if since if both $\mathrm{mor}(F)$ and $\mathrm{mor}(G)$ respect source and target maps, so does $\mathrm{mor}(G \circ F)$. If $s(F(m)) = F(s(m))$ and $s(G(m)) = G(s(m))$ for any morphism m, then $s(G(F(m))) = G(s(F(m))) = G(F(s(m)))$, and similarly for t. Similarly, $G \circ F$ respects composition: given morphisms $f : x \to y$ and $g : y \to z$ in C, we have $F(g \circ f) = F(g) \circ F(f)$. But then, given two functors $F : C_1 \to C_2$ and $G : C_2 \to C_3$, we have that

$G \circ F(g \circ f) = G(F(g \circ f)) = G(F(g) \circ F(f)) = G \circ F(g) \circ G \circ F(f)$. So $G \circ F$ is a pair of set functions preserving source and target maps and composition - i.e. a functor.

So finally we want to show that $\circ$ preserves source, target, and vertical composition for natural transformations. First, consider the effect of $\circ$ on natural transformations: suppose we have functors, $F, F' : C_1 \to C_2$ and $G, G' : C_2 \to C_3$, and natural transformations $\alpha : F \to F'$ and $\beta : G \to G'$, then we should have a natural transformation $\alpha \circ \beta$ from $G \circ F$ to $G' \circ F'$. Applying these functors in four possible combitations to a morphism $f : x \to y$ we get a commuting cube:

$$
\begin{tikzcd}
G \circ F(x) \arrow[rr, "G\circ F(f)"] \arrow[dr, "\beta(F(x)"] \arrow[dd, "G(\alpha(x))"'] & & G \circ F(y) \arrow[dr, "\beta(F(y)"] \arrow[dd, "G'(\alpha(y))"] & \\
& G' \circ F(x) \arrow[rr, "G'\circ F(f)"] \arrow[dd, "G'(\alpha(x))"] & & G' \circ F(y) \arrow[dd, "G'(\alpha(y))"] \\
G \circ F'(x) \arrow[rr, "G\circ F'(f)"] \arrow[dr, "\beta(F'(x))"] & & G \circ F'(y) \arrow[dr, "\beta(F'(y)"] & \\
& G' \circ F'(x) \arrow[rr, "G'\circ F'(f)"] & & G' \circ F'(y)
\end{tikzcd}
\tag{5}
$$

The squares of this cube all commute, by naturality of α and β and the fact that F and G' are functors. But then, this gives us, for each x, a map $G'(\alpha(x)) \circ \beta(F(x)) = \beta(F'(x)) \circ G(\alpha(x))$ from $G \circ F(x)$ to $G' \circ F'(x)$ and that these satisfy the commutation relation for a natural transformation. So we call this $\beta \circ \alpha(x)$. It is clear that $\circ$ preserves source and target maps for natural transformations, and vertical composition (which gives a commuting parallelepiped in the same way).

So indeed **Cat** *is a 2-category. QED*