COURSE NOTES ON QUANTIZATION AND COHOMOLOGY, WINTER 2007

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1. Preface

These are lecture notes taken at UC Riverside, in the Tuesday lectures of John Baez's Quantum Gravity Seminar, Winter 2007. The notes were taken by Apoorva Khare. Figures were prepared by Christine Dantas based on handwritten notes by Derek Wise. You can find the most up-to-date version of all this material here:

`http://math.ucr.edu/home/baez/qg-winter2007/`

These notes are a continuation of the Fall 2006 notes, available here:

`http://math.ucr.edu/home/baez/qg-fall2006/`

If you see typos or other problems with any of these notes, please let John Baez know (`baez@math.ucr.edu`).

2. Jan 16, 2007: Schrödinger's equation

<table>
<tr><th>Classical ($p = 1$)</th><th>Quantum ($p \geq 0$)</th></tr>
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(1) **Lagrangian Mechanics.** The path between two points (t_i, q_i) is $\gamma \in P_{q_0 \to q_1} = P_{q_0 \to q_1} Q$ satisfying $\delta S(\gamma) = 0$. Here,

- Q is the configuration space,
- $P_{q_0 \to q_1}$ is the path space,
- $\{\gamma : [t_0, t_1] \to Q : \gamma(t_i) = q_i\}$,
- $L : TQ \to \mathbb{R}$ is the Lagrangian, and
- $S(\gamma) := \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t))\ dt$ is the action.

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(1) **Lagrangian Mechanics.** The position of a particle at time t is given by a *wavefunction* $\psi_t : Q \to \mathbb{R}$. Moreover, the amplitudes $|\psi_t|$ form probability densities, so $\psi_t \in L^2(Q)$ for all t.
To consider "paths", we now have to integrate over all initial points and all paths ending at q_1. Thus, $\psi_{t_1}(q_1)$ now equals

$$\int_{q_0 \in Q} \int_{P_{q_0 \to q_1}} e^{iS(\gamma)/\hbar} \psi_{t_0}(q_0)(\mathscr{D}\gamma)\ dq_0$$

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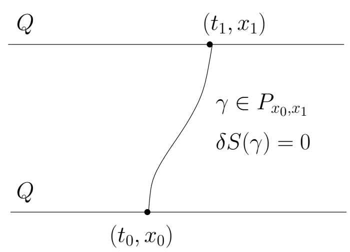

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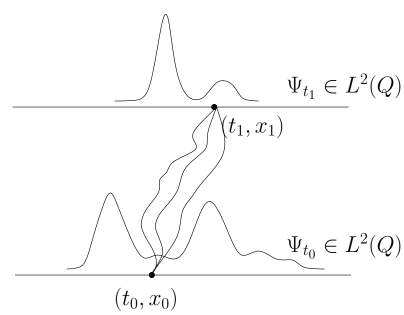

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(2) **Hamiltonian Mechanics.** We now have the phase space $X = T^*Q$ (or any symplectic manifold), with $H : X \to \mathbb{R}$ the Hamiltonian function. (In local coordinates (p_i, q^i), and using the Legendre transform $\lambda : TQ \to T^*Q$, we have $H = p_i \dot{q}^i - L \circ \lambda$.)
Then $\widetilde{\gamma}(t) := (q(t), p(t)) \in X$ satisfies Hamilton's equations:
$\frac{d}{dt}\widetilde{\gamma}(t) = v_H(\widetilde{\gamma}(t))$
where v_H is the Hamiltonian vector field, with $dH = \omega(v_H, -)$, where $\omega = -d\alpha$ is the symplectic structure.

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(2) **Hamiltonian Mechanics.** Each $\psi_t \in L^2(Q)$ satisfies *Schrödinger's equation*:
$\frac{d}{dt}\psi_t = \frac{1}{i\hbar}\widehat{H}(\psi_t)$
(look at it as $\psi : \mathbb{R} \to L^2(Q)$, say,) where $\widehat{H} : L^2(Q) \to L^2(Q)$ is a linear operator obtained (somehow!) by "quantizing" H.

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2.1. **Questions.** This chart raises lots of questions. (We mention a couple of them here.)

(1) How do you do "path-integrals" $\int_{P_{q_0 \to q_1}} \mathscr{D}\gamma$ over the path space?

Apparently, there is no meaning to the "measure" $\mathscr{D}\gamma$ (or it has not yet been found!), but there *is*, to $e^{iS(\gamma)/\hbar}\mathscr{D}\gamma$, at least in well-behaved cases. One such has been extensively studied by physicists: given a smooth finite-dimensional manifold Q, define

$$L(q, \dot{q}) = \frac{m}{2}||\dot{q}||^2 - V(q)$$

where we have the obvious kinetic and potential components. (If we replace i by -1 in the case of the *harmonic oscillator* $V(q) = |q|^2$, then the expression above, namely, $e^{-S(\gamma)/\hbar}\mathscr{D}\gamma$ is the *Wiener measure.*)

Digression on complete Riemannian manifolds: Note that to define the above Lagrangian, we need additional assumptions on Q, V. Namely, the kinetic component needs a metric, so we assume that the manifold is Riemannian. Moreover, we want "closed" manifolds, so that particles "don't fall off the edge" - so we assume that Q is complete as a metric space. Here, we have

Definition 2.1. Given a connected Riemannian manifold M,

(a) M is a metric space if we set the distance between points $m_0, m_1 \in M$ to be the *Riemannian distance*:

$$d(m_0, m_1) := \inf_{\gamma \in P_{m_0 \to m_1} M} |\gamma|$$

where $P_{m_0 \to m_1} M$ is the set of all *piecewise regular* curves (i.e. $\dot{\gamma}(t)$ is zero or undefined at most at finitely many points) from m_0 to m_1, and given any (smooth) parametrization $\gamma : [t_0, t_1] \to M$, we have its (parametrization-independent) *length*

$$|\gamma| := \int_{t_0}^{t_1} |\dot{\gamma}(t)| \; dt$$

Moreover, the metric topology is the same as the manifold topology.

(b) M is *geodesically complete* if every maximal geodesic is defined for all $t \in \mathbb{R}$.

Then (for "completeness' sake" ⌣̈!) we have the following result, that tells us a consequence of completeness:

Theorem 2.2 (Hopf-Rinow Theorem)**.** *A connected Riemannian manifold is complete (as a metric space) if and only if it is geodesically complete.*

Moreover, this is if and only if any two points can be joined by a geodesic - which is why this is relevant to us.

Back to our discussion: We thus define $L(q, \dot{q})$ as above, using the assumption that Q is a connected complete Riemannian manifold (to keep our particle from "falling off the edge"), and $V : Q \to \mathbb{R}$ should be smooth and bounded below (again to keep our particle from acquiring a lot of kinetic energy, and "shooting off to infinity" in finite time).

References. For the basic ideas, try Feynman and Hibbs, *Quantum Mechanics and Path Integrals*.

For mathematical rigor, try Barry Simons' *Functional Integration and Quantum Physics*.

(2) How do we get the Hamiltonian operator $\widehat{H} : L^2(Q) \to L^2(Q)$ from the Hamiltonian function $H : T^*Q \to \mathbb{R}$?

In some cases, it is easy to write down $\widehat{H}$, e.g. under the same assumptions we made while discussing the previous question:

$$H(q,p) = \frac{|p|^2}{2m} + V(q)$$

(where H is a connected complete (finite-dimensional) Riemannian manifold, and V is smooth and bounded below). In this situation, Schrödinger wrote:

$$\widehat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \mathrm{mult}_V$$

where $\nabla^2 := \overrightarrow{\nabla} \cdot \overrightarrow{\nabla}$ is the *Laplacian* (and $\overrightarrow{\nabla}$ the gradient). This sends a square-integrable function f on Q to

$$f \mapsto -\frac{\hbar^2}{2m}(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f) + f \cdot V$$

(where we compute the first term using the Riemannian metric). This is often written simply as $-\frac{\hbar^2}{2m}\nabla^2 + V$. Schrödinger got this by guessing the quantization rule: $p \mapsto \frac{\hbar}{i}\nabla$. (There were physical motivations for such a guess - namely, results for waves.) This yields:

$$\frac{|p|^2}{2m} = \frac{(p,p)}{2m} \mapsto -\frac{\hbar^2}{2m}\nabla^2$$

2.2. **Motivating geometric quantization.** Ideally, we would like

- a method to get $\widehat{H}$ from more general H.
- to use our assumptions (on Q and V) to show "good" properties of $\widehat{H}$.

For instance, if $A : K \to K$ is a self-adjoint operator on a Hilbert space K, then $e^{iAt} : K \to K$ is well-defined and unitary (preserves the inner product), and defining $\psi_t := e^{iAt}(\psi_0)$, we get

$$\frac{d}{dt}\psi_t = iA\psi_t$$

This is why we need the Schrödinger operator to be a self-adjoint operator in a Hilbert space - primarily for obtaining solutions to Schrödinger's equation! Which is what motivated Von Neumann and others to come up with the theory of Hilbert spaces and self-adjoint operators on them in the 20th century. Eventually, Kato and Rellich showed that the $\widehat{H}$ in our setup above, is indeed self-adjoint.

But we would like a much more systematic theory of "quantizing" functions $H : T^*Q \to \mathbb{R}$ and getting operators $\widehat{H} : L^2(Q) \to L^2(Q)$.

Even better, can we handle the case when the phase space X isn't T^*Q? (So we do not have Q - what would we replace $L^2(Q)$ by?)

This leads us to "geometric quantization". For more on this, try
`http://www.math.ucr.edu/home/baez/quantization.html`
Then try Sniatycki's book.

A lot of cohomology comes into the game - starting with the fact that $[\omega] \in H^2(X, \mathbb{R})$ must come from an *integral* cohomology class, i.e. $[\omega] \in \operatorname{im} \varphi$, where $\varphi : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$.

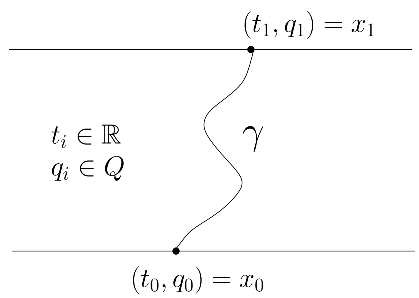
3. Jan 23, 2007: Categorification

Besides the "obvious" questions raised by the chart presented last time, there's a bigger question: **What's really going on?** Is the "quantization" some arbitrary trick (that nature has settled on), or does it have some deeper meaning? Let's try to dig deeper!

3.1. **A secret functor.**

What sort of entity is the action? Recall: in one approach, we have a configuration space Q, and then the action is a function $S : P_{(t_0,q_0)\to(t_1,q_1)} \to \mathbb{R}$, where if we denote $x_i := (t_i, q_i) \in \mathbb{R} \times Q$, then the domain is more precisely written as

$$P_{x_0\to x_1} := \{\gamma : [t_0, t_1] \to Q : \gamma(t_i) = q_i\}$$



But it's much deeper than this! Note that the action respects *composition of paths*: given $\gamma_i \in P_{x_{i-1}\to x_i}$, we can concatenate them to get a path $\gamma_1\gamma_2 \in P_{x_0\to x_2}$, and then

$$S(\gamma_1\gamma_2) = S(\gamma_1) + S(\gamma_2)$$

What this secretly means is that *the action is a functor*, from some category $\mathscr{P}$ with

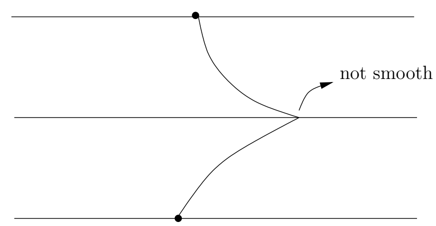
- $x = (t, q) \in \mathbb{R} \times Q$ as objects
- given objects x_0, x_1, paths $\gamma \in P_{x_0\to x_1}$ as morphisms (so each $P_{x_0\to x_1}$ is a Hom-space in $\mathscr{P}$)
- composition of morphisms given by concatenation of paths (this is indeed associative since we already have t in the "x-data", and don't need to reparametrize from $[0, 1] \to Q$);

to some category $\mathbb{R}$ with

- one object, $*$
- all real numbers as morphisms
- composition of morphisms is just addition (a group is just a one object category, with all morphisms invertible).

Technical remarks:

(1) We cannot use all possible paths in defining $\mathscr{P}$ (since the action and Lagrangian involved the derivative $\dot{\gamma}$), nor only smooth paths (since composing paths might result in "corners"):



[figure: a concatenation of paths γ_1, γ_2 with a "cusp"-like point of non-differentiability at the common point $x_1 = (t_1, q_1)$]

We could use piecewise smooth paths, though.

(2) Alternatively, we could have specified both the position and velocity (to avoid "corners"), but note that the calculus of variations involved in picking out an "extremal" path automatically chooses these by itself! So if we specified both postiion and velocity, then an extremal path might not exist. (Physically, we don't want our particle/path to have "too much inherent data".)

(3) Note that the above just talks of the action; we'll now see how to *use* this categorical approach for classical and quantum mechanics. Thus, (in quantum mechanics) a path does not represent a particle's (time) evolution, only (the integral over) the entire path space does! So, even though we do not know the positiion and momentum at the same time for a *particle*, we do know it for each specific path!

3.2. **Bringing in arbitrary categories.** So, can we do classical and quantum mechanics starting with any functor $S : \mathcal{C} \to \mathbb{R}$ (where now $\mathcal{C}$ is *any* category), with

- "configurations" as objects, and
- "paths" as morphisms?

To see this, we ask: how did we use $S : \mathcal{C} \to \mathbb{R}$ in the chart last time?

Classically, we can do one of two things:
(1) We "criticize" it - i.e. for each $x, y \in \mathcal{C}$, we look at $S : \mathrm{Hom}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathbb{R}$ and seek critical points, i.e. $\gamma \in \mathrm{Hom}(x, y)$ with $dS(\gamma) = 0$.

[figure: Q (or $\mathcal{C}$) on both sides, with $\gamma : x \to y$ and $\delta S(\gamma) = 0$]

This only makes sense if each set $\mathrm{Hom}(x, y)$ is a manifold or a more general "infinite-dimensional manifold" (e.g. a space of piecewise smooth paths in a manifold Q), and S is differentiable.

This is addressed by the theory of smooth categories and smooth functors, cf. `http://www.math.ucr.edu/home/baez/2conn.eps`.

($1'$) We "minimize" it - i.e. for each $x, y \in \mathcal{C}$, we seek $\gamma \in \mathrm{Hom}(x, y)$ that minimizes $S(\gamma)$. (We might need each Hom-space to be a topological space (possibly compact!), and S "continuous".)

Remarks:

(1) For both (1) and ($1'$), the issues of *existence* and *uniqueness* of a γ criticizing/minimizing the action, are very important.

(2) Case ($1'$) is closer to quantum mechanics, which we can see if we study *Hamilton's principal function*: given $x, y \in \mathcal{C}$, define

$$Z(x, y) := \inf_{\gamma \in \mathrm{Hom}(x,y)} S(\gamma)$$

(assuming this infimum exists). In classical mechanics, this is very important; we get the *Hamilton-Jacobi equations* by differentiating $Z(x, y)$ with respect to x or y, and working out the answer. These are the classical analogue(s) of Schrödinger's equation.

Now consider the **quantum case**.

(2) We integrate its exponential - i.e. for each x, y, we compute a *(transition) amplitude*

$$Z_\hbar(x, y) := \int_{\gamma \in \mathrm{Hom}(x,y)} e^{iS(\gamma)/\hbar} \, \mathscr{D}\gamma$$

For this, we want each set $\mathrm{Hom}(x, y)$ to be a measure space (or generalized measure space), and $e^{iS(\gamma)/\hbar}$ must be integrable.

In this case, we can get Schrödinger's equation by fixing x and differentiating $Z_\hbar(x, y)$ with respect to y (or vice versa), and working out the answer.

Now compare cases ($1'$) (i.e. $Z(x, y)$) and (2) (i.e. $Z_\hbar(x, y)$): the classical and quantum cases.

Classical	Quantum
(a) $S(\gamma) \in \mathbb{R}$.	(a) $e^{iS(\gamma)/\hbar} \in U(1) \subset \mathbb{C}$ (because integrating this gives us a number in $\mathbb{C}$, not $U(1)$).
(b) We take the infimum (i.e. minimum).	(b) We take the integral (i.e. sum).
(c) The group morphism is addition: $S(\gamma_1\gamma_2) = S(\gamma_1) + S(\gamma_2)$.	(c) The group morphism is multiplication: $e^{iS(\gamma_1\gamma_2)/\hbar} = e^{iS(\gamma_1)/\hbar} \cdot e^{iS(\gamma_2)/\hbar}$.

Thus, we now ask: How are both of these, special cases of some "prescription" for getting physics out of the action?

4. Jan 30, 2007: Physics is rigged!

4.1. **The analogous viewpoints.** From last time, we've seen a "big analogy" between classical and quantum mechanics of point particles. As we saw, when we minimize/integrate, the identity for this operation in $\mathbb{R}/\mathbb{C}$ (respectively) is $+\infty/0$. Thus, we need to replace the $\mathbb{R}$ used above, by $\mathbb{R}^{min} := \mathbb{R} \cup \{+\infty\}$ throughout.

Moreover, since we add the actions of paths whenever we compose (in the classical case), we need to expand the addition operation to all of $\mathbb{R}^{min}$. This is done by: $+\infty + x = +\infty$ for all x, and $(\mathbb{R}^{min}, +, 0)$ is a commutative monoid.

We can now say more about the above analogy:

	Classical mechanics of point particles	Quantum mechanics of point particles
(a) We start with the	action $S(\gamma) \in \mathbb{R}^{min}$.	amplitude $e^{iS(\gamma)/\hbar} \in \mathbb{C}$.
(b) What we do:	take the infimum/minimum.	take the integral/sum.
(c) Operational identity:	$+\infty$ - so $(\mathbb{R}^{min}, \min, +\infty)$ is a commutative monoid.	0 - so $(\mathbb{C}, +, 0)$ is an abelian group.
(d) When we compose paths, we	add actions.	multiply amplitudes.
(e) Operational identity:	0.	1.
(f) Overall:	$(\mathbb{R}^{min}, \min, +\infty, +, 0)$ is a commutative *rig*.	$(\mathbb{C}, +, 0, \cdot, 1)$ is a commutative ring.

Remark 4.1.

(1) We'll see later, how the picture on the right is really a one-parameter family (one for every $\hbar$), and we "go to the classical case" as $\hbar \to 0$.
(2) A <u>rig</u> is a "<u>ring</u> without <u>n</u>egatives", i.e. a commutative monoid under addition, and a monoid under multiplication, satisfying left/right distributive laws. (In particular, every ring is a rig.)
(3) $\mathbb{R}^{min}$ is a rig that satisfies $x \min x = x$ for all x. Thus, it satisfies an "idempotence" property, sort of the opposite of the usual "cancellational" addition that we see in a group ($a * b = a * c \Rightarrow b = c$.)

4.2. **Switching between the classical and quantum viewpoints.** In the rest of this lecture, we'll go back and forth between the classical and quantum viewpoints. For instance,

(1) In classical mechanics, action is a *functor* $S : \mathcal{C} \to \mathbb{R}^{min}$, where $\mathcal{C}$ is any category (whose objects - resp. morphisms - are called

"configurations" - resp. "paths"), and $\mathbb{R}^{min}$ is a category with one object whose morphisms are $x \in \mathbb{R}^{min}$ and composition is addition (i.e. the "multiplication" in the rig): $S(\gamma_1\gamma_2) = S(\gamma_1) + S(\gamma_2)$.

Let's now "quantize" this statement!

In quantum mechanics, amplitude is a functor $e^{iS(\cdot)/\hbar} : \mathcal{C} \to \mathbb{C}$, where $\mathcal{C}$ is as above, and $\mathbb{C}$ is a category with one object whose morphisms are $x \in \mathbb{C}$ and composition is $\cdot$: $e^{iS(\gamma_1\gamma_2)/\hbar} = e^{iS(\gamma_1)/\hbar} \cdot e^{iS(\gamma_2)/\hbar}$.

Remark 4.2. The amplitude functor is called $e^{iS(\cdot)/\hbar}$ more for sentimental reasons here, than out of any connection with S itself.

(2) We now present an **example**, wherein we "translate" a concept (over to the "other side") using this analogy. However, unlike earlier, we now start on the "quantum side"!

Say $\mathcal{C} = \mathscr{P}$ (as in the last class), with

- objects as points $x = (t, q) \in \mathbb{R} \times Q$, where Q is some "configuration space" (manifold),
- morphisms $\gamma : x_0 = (t_0, q_0) \to x_1 = (t_1, q_1)$ are paths : $[t_0, t_1] \to Q$ so that $\gamma(t_i) = q_i$.

In the **quantum case**, a wavefunction $\psi : Q \to \mathbb{C}$ tells us the amplitude for a particle to be at $q \in Q$. Of course, this is not a "good" thing because we have Q (and not $\mathbb{R} \times Q$), and this is not really among the objects or morphisms of our category $\mathcal{C}$.

But now, we can bring in $\mathcal{C}$ as follows: We describe the time-evolution of ψ by

$$\psi(t_1, q_1) = \int_{q_0 \in Q} \int_{P_{q_0 \to q_1}} e^{iS(\gamma)/\hbar} \psi_{t_0}(q_0)(\mathscr{D}\gamma)\ dq_0$$

The **classical** analogue of a wavefunction is a known entity in physics; let's call it ψ_c for short. (Just as $S(\cdot) \mapsto e^{iS(\cdot)/\hbar}$, we really should have ψ in the quantum case coming from $-i\hbar \ln \psi$, but we just use ψ_c.)

Thus, classically, $\psi_c : Q \to \mathbb{R}^{min}$ tells us the action for a particle to be at $q \in Q$. By our analogy, it should evolve in time as follows:

$$\psi_c(t_1, q_1) = \inf_{q_0 \in Q} \inf_{\gamma:[t_0,q_0] \to [t_1,q_1]} (S(\gamma) + \psi_c(t_0, q_0))$$

Remark 4.3.

(a) Note that t_0 is fixed.

(b) The $\mathscr{D}\gamma \cdot dq_0$ *now* just gives information / identifies the space over which we integrate / minimize.

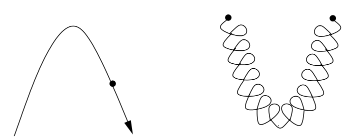
(c) If we imagine $\psi_c(t_0, q_0)$ as the cost to "start a trip" at $q_0 \in Q_0$ at time t_0, and $S(\gamma)$ as the cost of the trip γ, this formula tells us that $\psi_c(t_1, q_1)$ is the cheapest price to be at $q_1 \in Q$ at time

t_1. (Note that there may be many different "equally cheap" ways to get there!)

(3) In the quantum case, you can go ahead and use the path integral to compute $\dfrac{d}{dt}\psi$ - you get Schrödinger's equation.
In the classical case, you get the *Hamilton-Jacobi equations*.

4.3. **Wick rotation and a spring in imaginary time (revisited).** Last quarter, we saw that the *dynamics of point particles* is analogous to the *statics of strings*. This analogy involves *Wick rotation*, namely, the substitution $t \mapsto -it$. (Think of how "rotating" clockwise by $90°$ amounts to multiplying by $-i$ in the complex plane.)

For example, consider a rock and a spring in a gravitational field:



The action for the rock is

$$S = \int_{t_0}^{t_1} \left[\frac{m}{2}\ \dot{q}\cdot\dot{q} - V(q(t))\right]\ dt$$

which, under Wick rotation, becomes (note that $\dot{q} = \frac{d}{dt}q$, so we have to bring in -1 from the $\dot{q}\cdot\dot{q}$ now)

$$\int_{-it_0}^{-it_1} \left[-\frac{m}{2}\ \dot{q}\cdot\dot{q} - V(q(-it))\right]\ d(-it)$$

for the spring. This was what we referred to as the "spring in imaginary time" early last quarter! In this (second/spring) case, we write it as the *energy* (cancelling various powers of $(-i)$ and renaming variables):

$$E = -iS = \int_{t_0}^{t_1} \left[\frac{m}{2}\ \dot{q}\cdot\dot{q} + V(q(t))\right]\ dt$$

since this is the energy of the spring, where m is now the *tension* (spring constant), $\dot{q}$ refers to how "stretched" the spring is, and the energy is the sum of the "tension energy" and the gravitational (potential) energy.

Next time: The rig $\mathbb{R}$ (not $\mathbb{R}^{min}$) comes into play!

5. Jan 23, 2007: Statistical mechanics and deformation of rigs

We saw last time that the classical mechanics (dynamics) of particles becomes the classical statics of strings by doing the substitutions

$$t \mapsto -it, \qquad S \mapsto iE$$

Minimizing action now becomes minimizing energy.

What does the quantum mechanics (dynamics) of particles become when we do these substitutions?

5.1. **Statistical mechanics "quantizes" strings.**

In quantum mechanics, the relative amplitude for a particle to trace out a path is $e^{iS()/\hbar}$. In "statistical" *mechanics* (really *thermal statics* - classical statics but with nonzero temperature T) - and let us not even talk of thermo*dynamics* now! - the *relative* probability for a system to be in configuration of energy E is

$$e^{-E/kT}$$

where k is *Boltzmann's constant* (a conversion factor between energy and temperature).

Note that we have

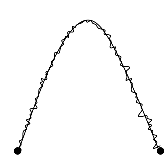
$$e^{iS/\hbar} \mapsto e^{-E/kT}$$

if we do the substitutions

$$S \mapsto iE, \qquad \hbar \mapsto kT$$

(or should we really have $S \mapsto iE/T$, because $\hbar$ is a constant? But we also want to get to classical mechanics by $\hbar \to 0$, or $T \mapsto 0$!).

This makes some sense since $\hbar$ measures how big "quantum fluctuations" are:

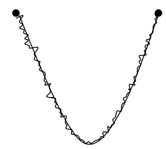


"quantum version of a thrown rock"

(Thus, all paths "far" from the path of least action cancel one another out, and only the "nearby" paths contribute.)

while kT measures how big "thermal fluctuations" are:

(Moreover, the relative quantities need to be normalized to give the amplitude/probability.)



"hanging spring at
nonzero temperature"

5.2. **A family of rigs via the Boltzmann map.** Henceforth I'll set $k = 1$ and use the substitution $\hbar \mapsto T$. Note that

- $e^{iS/\hbar} \in \mathbb{C}$, the rig of relative amplitudes, and
- $e^{-E/T} \in \mathbb{R}^+$, the rig of relative probabilities.

(Here, $\mathbb{R}^+ = ([0, \infty), +, 0, \cdot, 1)$.) In short, we have

Particles ($p = 1$)	Strings ($p \geq 0$)
(1) Classical dynamics - we deal with action $S \in \mathbb{R}^{min}$	(1) Classical statics - we deal with energy $E \in \mathbb{R}^{min}$ (a "different/imaginary" $\mathbb{R}^{min}$)
(2) Quantum dynamics - we deal with relative amplitude $e^{iS/\hbar} \in \mathbb{C}$	(2) Thermal statics - we deal with relative probability $e^{-E/T} \in \mathbb{R}^+$

We note that to go from the first column to the second, we use Wick rotation:

$$t \mapsto -it, \qquad S \mapsto iE, \qquad \hbar \mapsto T$$

We'd like to understand how quantum mechanics reduces to classical mechanics as $\hbar \to 0$, but it's easier to understand how thermal statics reduces to classical statics as $T \to 0$.

To do this, we'll formulate thermal statics using E instead of $e^{-E/T}$: for any $T > 0$, we consider the *Boltzmann map* $\beta_T : \mathbb{R}^{min} \to \mathbb{R}^+$:

$$E \mapsto e^{-E/T} \qquad (+\infty \mapsto 0)$$

This isn't a rig homomorphism, just a one-to-one and onto function. So, we'll pull back the rig structure on $\mathbb{R}^+$ to (the *set*) $\mathbb{R}^{min}$ via β_T, and get a rig $\mathbb{R}^T$.

As a set, $\mathbb{R}^T$ is just $[0, \infty)$, but now it's a rig with

$$\begin{aligned} a +_T b &:= \beta_T^{-1}(\beta_T(a) + \beta_T(b)) \\ 0_T &:= \beta_T^{-1}(0) \\ a \cdot_T b &:= \beta_T^{-1}(\beta_T(a)\beta_T(b)) \\ 1_T &:= \beta_T^{-1}(1) \end{aligned}$$

Homework. Work out $+_T, 0_T, \cdot_T, 1_T$ explicitly, and show that

$$\lim_{T \to 0} \beta_T^{-1}(\mathbb{R}^+) = \mathbb{R}^{min}$$

or, in other words, that

$$\lim_{T\to 0}(+_T, 0_T, \cdot_T, 1_T) = (\min, +\infty, +, 0)$$

So, the "topological rig" $\mathbb{R}^T$ converges to the topological rig $\mathbb{R}^{min}$ as $T \to 0$.

Also note that as $T \to +\infty$, all elements $\beta_T(a)$ converge to either 0 (if $a \in \mathbb{R}$) or 1 (if $a = +\infty$) - "impossible" or "possible" events respectively. (That is, when the going gets hot, everything that is possible appears equally likely!)

(We'd be very happy if this were to be in a different rig - the logic rig of truth values $(\{0,1\}, \vee, \wedge)$. But this is not so.)

5.3. **The analogous situation for quantization.** The moral of the above analysis is that "thermal statics reduces to classical statics as $T \to 0$"; in both cases we're really doing linear algebra over some rig, and $\mathbb{R}^T \to \mathbb{R}^{min}$ as $T \to 0$.

Alas, seeing classical mechanics as an $\hbar \to 0$ limit of quantum mechanics is harder, since

$$\beta_\hbar : \mathbb{R}^{min} \to \mathbb{C} \text{ sending } S \mapsto e^{iS/\hbar},\ +\infty \mapsto 0$$

is neither one-to-one nor onto, and its image is not a subrig (though it's closed under multiplication). So we can't pull the rig structure on $\mathbb{C}$ back to $\mathbb{R}^{min}$.

However, people do study quantization *indirectly* using the $\lim\limits_{T\to 0} \mathbb{R}^T = \mathbb{R}^{min}$ idea, which is called

- *tropical mathematics* (a really stupid term for the work of Brazilian mathematicians - "Arctic mathematics" would be better for $T \to 0$ math!)
- *idempotent analysis* (since $a \min a = a$ in $\mathbb{R}^{min}$)
- *Maslov dequantization* (in reference to how $T \to 0$ limit lets us study the $\hbar \to 0$ limit).

Solution to the homework. It is easy to see the following:

$$\begin{aligned}
\beta_T^{-1}(a) &:= -T\ln(a) \\
a +_T b &:= -T\ln(e^{-a/T} + e^{-b/T}) \\
0_T &:= -T\ln 0 = +\infty \\
a \cdot_T b &:= -T\ln(e^{-(a+b)/T}) = a + b \\
1_T &:= -T\ln 1 = 0
\end{aligned}$$

This means that there is only one limit left to verify: that of $a +_T b$ as $T \to 0$. So say $a \leq b$. Then we compute:

$$a +_T b = -T\ln(e^{-a/T}(1 + e^{(a-b)/T})) = -T\ln(e^{-a/T}) - T\ln(1 + e^{(a-b)/T})$$

The first term clearly equals a. Now denote $\alpha = e^{a-b}$. Then $\alpha \in (0,1]$. As $(1 >)T \to 0^+$, we see that $\alpha^{1/T} \in (0,1]$, whence $\ln(1+\alpha^{1/T})$ is bounded. Therefore, by the Pinching Theorem,

$$\lim_{T\to 0} a +_T b = a + \lim_{T\to 0^+} T \cdot \ln(1+\alpha^{1/T}) = a + 0$$

and we are done, since $a \min b = a$. □

6. Feb 13, 2007: An example of path integral quantization - I

We have a strategy for quantization, given any category $\mathcal{C}$ (of "configurations" and "paths") and a functor

$$S : \mathcal{C} \to (\mathbb{R}, +)$$

(the "action"). This gives a functor

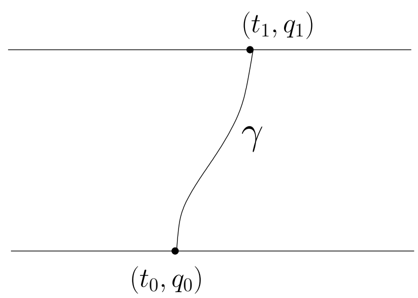
$$e^{iS/\hbar} : \mathcal{C} \to (\mathbb{C}, \cdot)$$

(the "amplitude"), and we compute the "transition amplitude" from any object $x \in \mathcal{C}$ to $y \in \mathcal{C}$ via

$$Z_\hbar(x, y) = \int_{\gamma : x \to y} e^{iS(\gamma)/\hbar} \mathscr{D}\gamma$$

which requires also that we have measures on hom-sets $\hom_{\mathcal{C}}(x, y)$.

6.1. **Example: free particle on the real line.** Let's do an example - the free particle on $\mathbb{R}$. Here, the objects of $\mathcal{C}$ form the set $\mathbb{R}^2 \ni (t, q)$, and morphisms $\gamma : (t_0, q_0) \to (t_1, q_1)$ are paths $\gamma : [t_0, t_1] \to \mathbb{R}$, so that $\gamma(t_i) = q_i$.



(This is the category $\mathscr{P}$ - with $Q = \mathbb{R}$ - that we introduced earlier, and then the morphism spaces $\{\gamma : x_0 \to x_1\}$ were called $P_{q_0 \to q_1}$.) Thus,

$$Z_\hbar((t_0, q_0), (t_1, q_1)) = \int_{P_{q_0 \to q_1}} e^{iS(\gamma)/\hbar} \mathscr{D}\gamma$$

where S is the action for a free (i.e. no potential) particle of mass m:

$$S(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t))\, dt$$

with Lagrangian given by

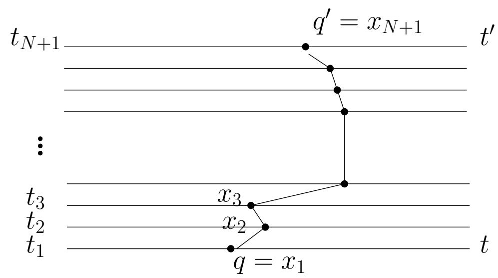
$$L(q, \dot{q}) = \frac{m}{2}\dot{q}^2$$

since there's no potential.

To do the path integral over *all* paths γ, we first integrate only over piecewise linear paths (and worry about "taking the limit" later). However, we first need a change of notation!

$$(t_0, q_0) \leftrightarrow (t, q), \qquad (t_1, q_1) \leftrightarrow (t', q')$$

We thus consider piecewise linear paths $\gamma : (t, q) \to (t', q')$



for some chosen times $t_1, \dots, t_{N+1}$, where

$$P_{\mathbf{t}} := \{t = t_1 < t_2 < \cdots < t_N < t_{N+1} = t'\}$$

is a (not necessarily regular) partition of $[t, t']$ (also called a *mesh*), and γ is piecewise linear on each subinterval.

To integrate over all these piecewise linear paths, we just integrate over $x_2, \dots, x_N \in \mathbb{R}$, where $x_j = \gamma(t_j)$.

Then we'll try to show that these integrals over (the space of) piecewise linear paths converge as the *norm* $||P_{\mathbf{t}}||$ of the partition goes to zero. To use another name for the norm, we'll see what happens as the *mesh spacing*

$$\max_j (t_{j+1} - t_j)$$

goes to zero.

6.2. **Doing the math.** But first, let's see what these integrals look like - let's compute one:

$$A_{P_{\mathbf{t}}} = A_{P_{\mathbf{t}}}((t, q), (t', q')) = \int_{\mathbb{R}^{N-1}} \exp\left(\frac{i}{\hbar}\int_t^{t'} \frac{m}{2}\dot{\gamma}(s)^2\ ds\right)\ dx_2 \dots dx_N$$

Actually, we need to rescale the Lebesgue measure by *normalizing factors*:

$$dx_j \mapsto \frac{dx_j}{c_j},\ c_j \in \mathbb{R}$$

where c_j depends on $t_{j+1} - t_j$ (for all j). We need these to get convergence as the mesh spacing goes to zero.

Question. The c_j's are chosen to make the math work out fine. (These are what one calls *(re?)normalizations.*) But what is the *physics* behind choosing them? (E.g. is it just analogous to rescaling in order to get the total probability to equal 1?) Many people would be very happy to know...

But γ is piecewise-linear, so on the jth piece $[t_j, t_{j+1}]$, we have

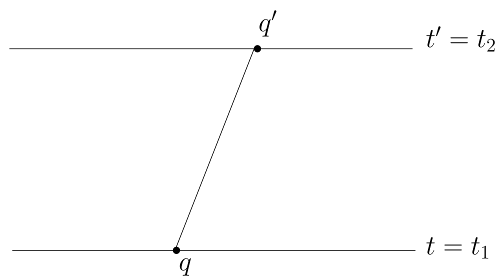
$$\dot{\gamma}(s) \equiv \frac{x_{j+1} - x_j}{t_{j+1} - t_j} = \frac{\Delta x_j}{\Delta t_j}$$

where $\Delta x_j = x_{j+1} - x_j,\ \Delta t_j = t_{j+1} - t_j$ for all j. Hence,

$$A_{P_{\mathbf{t}}} = \int_{\mathbb{R}^{N-1}} \exp\left(\frac{im}{2\hbar}\sum_{j=1}^{N}\left(\frac{\Delta x_j}{\Delta t_j}\right)^2 \Delta t_j\right)\ \frac{dx_2}{c_2}\cdots\frac{dx_N}{c_N}$$

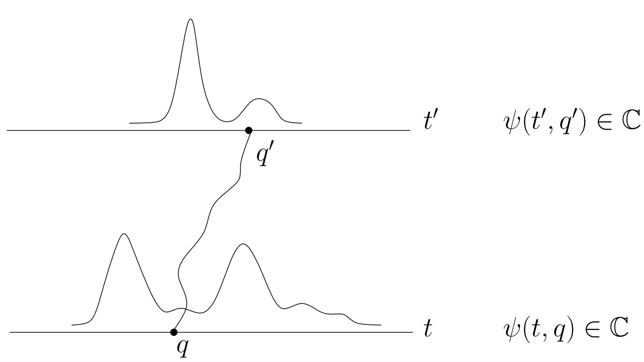
The crucial thing is that if we choose the c_j's correctly, $A_{P_{\mathbf{t}}}$ is actually *independent* of the mesh/partition $P_{\mathbf{t}}$ - so convergence is trivial!

In other words, we can actually compute $Z_\hbar((t,q),(t',q'))$ as an integral over *linear* paths



of which there is only one.

To prove that $A_{P_{\mathbf{t}}}$ is independent of the mesh $P_{\mathbf{t}}$, let's think instead about the rule for evolving a wavefunction ψ in time:



$$\psi_{t'}(q') = \int_{\mathbb{R}}\int_{\gamma:(t,q)\to(t',q')} e^{iS(\gamma)/\hbar}\psi_t(q)\ \mathscr{D}\gamma\ dq = \int_{\mathbb{R}} Z_\hbar((t,q),(t',q'))\psi_t(q)\ dq$$

When we approximate $Z_\hbar$ by integrating over piecewise linear paths, we get ($\widetilde{\psi}_t$ is the approximation, and we write out the exponentials in the order in which they are applied to the original wavefunction $\psi_t(q)$):

$$\begin{aligned}
&\widetilde{\psi}_{t'}(q') \\
= &\int_{\mathbb{R}^N} \exp\left(\frac{im}{2\hbar}\frac{(\Delta x_N)^2}{\Delta t_N}\right)\cdots\exp\left(\frac{im}{2\hbar}\frac{(\Delta x_1)^2}{\Delta t_1}\right)\psi_t(q)\ dq\ \frac{dx_2}{c_N}\cdots\ \frac{dx_N}{c_N} \\
= &\int_{\mathbb{R}^N} K(\Delta t_N, \Delta x_N)\cdots K(\Delta t_1, \Delta x_1)\psi_t(q)\ dq\ dx_2\cdots dx_N
\end{aligned}$$

where

$$K(\Delta t_j, \Delta x_j) := \frac{1}{c_j}\exp\left(\frac{im}{2\hbar}\frac{(\Delta x_j)^2}{\Delta t_j}\right)$$

is the "kernel".

Remark 6.1.

(1) We have to justify the third equality in

$$\begin{aligned}
\psi_{t'}(q') &= \int_{\mathbb{R}^N} Z_\hbar \cdot \psi_t(q) = \int_{\mathbb{R}^N} \lim_{||P_{\mathbf{t}}||\to 0} A_{P_{\mathbf{t}}} \cdot \psi_t(q) \\
&= \lim_{||P_{\mathbf{t}}||\to 0}\int_{\mathbb{R}^N} A_{P_{\mathbf{t}}} \cdot \psi_t(q) = \lim_{||P_{\mathbf{t}}||\to 0} \widetilde{\psi}_{t'}(q')
\end{aligned}$$

But if $A_{P_{\mathbf{t}}}$ is independent of $P_{\mathbf{t}}$, then the equality becomes obvious! Moreover, we shall see next time that this *does*, indeed, hold.

(2) To show that $A_{P_{\mathbf{t}}}$ is independent of $P_{\mathbf{t}}$, we just need to show that $\widetilde{\psi}_{t'}(q')$ is independent of $P_{\mathbf{t}}$ for all ψ_t. (This is exactly like showing that $\int fg = 0\ \forall g \Rightarrow f = 0$, for then we can choose various delta-functions for ψ_t, and get that $\widetilde{\psi}_{t'}(q) = 0$ for all q.)

(3) To show independence from $P_{\mathbf{t}}$, it is enough to show that if we refine the mesh/partition by one point, we get the *same* kernel. This is because if we are then given A_P, A_Q for partitions P, Q, then they are both equal to $A_{P\cup Q}$.

Thus, it is now enough to show that

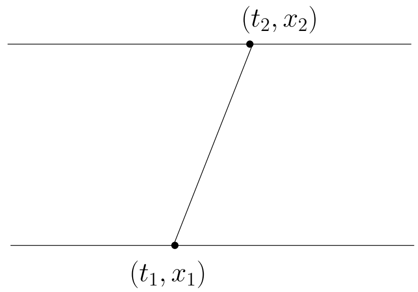
$$K(t_3 - t_1, x_3 - x_1) = \int_{x_2\in\mathbb{R}} K(t_3 - t_2, x_3 - x_2)K(t_2 - t_1, x_2 - x_1)\ dx_2$$

[figure: γ is a line, but we introduce an intermediate time t_2 and a non-linear, piecewise-linear curve from t_1 to t_3]

(4) Note that this is a proof where the c_j's are unknown! So we will in fact use the proof to determine the c_j's (so that the above condition holds). Moreover, we need c_j to be a function of at most the time - and as we will see next time, c_j actually depends only on Δt_j!

7. Feb 20, 2007: An example of path integral quantization - II

Last time, we considered path integral quantization for a free particle on a line. (This is “exactly solvable”.) We claimed that the exact answer could be found by considering just *one* path - the straight line path



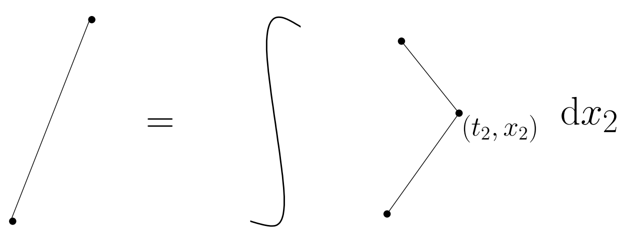
- and essentially evaluating $e^{iS/\hbar}$ to go from (t, x) to (t', x'):

$$K(\Delta t, \Delta x) = \frac{1}{c(\Delta t)} \exp\left(\frac{im}{2\hbar}\frac{(\Delta x)^2}{\Delta t}\right)$$

where $c = c(\Delta t)$ is a normalizing factor (that we have yet to determine). We saw that to prove this, we just need to check

$$K(t_3 - t_1, x_3 - x_1) = \int_{x_2 \in \mathbb{R}} K(t_3 - t_2, x_3 - x_2) K(t_2 - t_1, x_2 - x_1)\, dx_2 \quad (\bigstar)$$

i.e., in pictures,



7.1. **Time-evolution operators.** To prove $(\bigstar)$ (for some $c(\Delta t)$), we could just do the integral - but this is too annoying (for JB!). So instead, we'll take a more conceptual route:

Consider the operator $U(t)$ which describes one step of time evolution (via straight-line paths only):

[figure: wavefunctions at times t_1, t_2; the endpoint is fixed at (t_2, x_2), but there are lots of lines leading to it, all from potential starting points (t_1, x_1)]

$$(U(t_2 - t_1)\psi_{t_1})(x_2) = \int_{\mathbb{R}} K(t_2 - t_1, x_2 - x_1)\psi_{t_1}\ dx_1$$

This tells us the wavefunction at time t_2 in terms of the wavefunction at time t_1 (i.e. ψ_{t_1}), as an integral over straight-line paths from (t_1, x_1) to (t_2, x_2).

Remark 7.1. Thus, *kernels* really are functions of two variables that are like matrices, but with entries indexed by a *continuous* set. Integrating against a kernel is like matrix multiplication - see the last two equations above!

In these (above) terms, ($\bigstar$) simply says

$$U(t_3 - t_1)\psi_{t_1} \equiv U(t_3 - t_1)U(t_2 - t_1)\psi_{t_1}\ \forall \psi_{t_1},\ \forall t_i \in \mathbb{R}$$

or equivalently,

$$U(t + s) \equiv U(t)U(s)\ \forall t, s \in \mathbb{R}$$

(One way is clear: integrate the second identity against any ψ_{t_1} to get the first. Conversely, the familiar principle of $\int fg = 0\ \forall g \Rightarrow f \equiv 0$ suggests that we use delta-functions in place of ψ_{t_1} to get the second equation at all points.)

7.2. **Bringing in the Hamiltonian.** We would know this if we could write

$$U(t) \equiv \exp\left(\frac{-itH}{\hbar}\right) = \exp\left(\frac{t}{i\hbar}H\right)$$

for some operator H, since we then get

$$\exp\left(\frac{-i(t+s)H}{\hbar}\right) \equiv \exp\left(\frac{-itH}{\hbar}\right) \cdot \exp\left(\frac{-isH}{\hbar}\right)$$

So we will show that $U(t) = \exp((t/i\hbar)H)$ for

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

(as Schrödinger had obtained). Here, H is the *Hamiltonian* for the free particle.

When we show this, we'll see that if $\psi_t = U(t)\psi_0$, then ψ_t will satisfy Schrödinger's equation:

$$\frac{d}{dt}\psi_t = \frac{d}{dt}\left[\exp\left(\frac{t}{i\hbar}H\right)\psi_0\right] = \frac{1}{i\hbar}H\left(e^{(t/i\hbar)H}\psi_0\right) = \frac{1}{i\hbar}H\psi_t$$

So we just need to check that $(e^{(t/i\hbar)H}\psi)(x)$ is the same as

$$(U(t)\psi)(x) = \int_{y\in\mathbb{R}} K(t, x-y)\psi(y)\ dy$$

Since both sides depend linearly on ψ, and both sides are translation-invariant, it suffices to check this for $\psi = \delta$, the Dirac-delta at the origin. In other words,

$$\left(e^{(t/i\hbar)H}\delta\right)(x) = K(t,x) = K(t-0, x-0)$$

Remark 7.2.

(1) To make this rigorous, we need to
- introduce some topology on the space of wavefunctions,
- show that in this space, the set of finite linear combinations of delta functions is "dense" in this topology, and
- show that $\exp(\frac{t}{i\hbar}H), U(t)$ are "continuous" on this space in this topology.

(2) Note that in this last equation,
- the left side is the Hamiltonian way of computing the amplitude for a particle to end up at position x at time t if it (definitely!) starts at the origin at time 0; and
- the right side is the Lagrangian way to compute the same thing - but integrating only over straight-line paths!

7.3. **Computing the normalizing factors.** Since we (almost) know one side - $K(t,x)$ - we'll just compute the other side of the above equation. We'll use the *Fourier Transform*

$$\widehat{\psi}(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx}\psi(x)\ dx \qquad (k\in\mathbb{R})$$

and its *inverse*

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx}\widehat{\psi}(k)\ dk$$

(Here, "k" stands for momentum.)

Next, we need a couple of small computations. Firstly, how do differentiation and the Fourier transform interact? We use integration by parts to compute:

$$\widehat{\psi'}(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx}\frac{d}{dx}\psi(x)\ dx = ik\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx}\psi(x)\ dx = ik\widehat{\psi}(k)$$

so that for $H = -\dfrac{\hbar^2}{2m}\dfrac{d^2}{dx^2}$, we get (using the Taylor series expansion)

$$\begin{aligned} \widehat{e^{(t/i\hbar)H}\psi}(k) &= \widehat{e^{\frac{-t}{i\hbar}\frac{\hbar^2}{2m}\frac{d^2}{dx^2}}\psi}(k) = \sum_{n\geq 0}\frac{1}{n!}\left(\frac{it\hbar}{2m}\right)^n \widehat{\frac{d^{2n}}{dx^{2n}}\psi}(k) \\ &= \sum_{n\geq 0}\frac{1}{n!}\left(\frac{it\hbar\cdot i^2k^2}{2m}\right)^n \widehat{\psi}(k) = e^{-itk^2\hbar/2m}\widehat{\psi}(k) \end{aligned}$$

To simplify future computations, let's pick units where $\hbar = 1$ and $m = 1$, to lessen the mess. So we now know that $e^{(t/i\hbar)H}$ now becomes e^{-itH}, and we compute:

$$\widehat{e^{-itH}\delta}(k) = e^{-itk^2/2}\widehat{\delta}(k) = e^{-itk^2/2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx}\delta(x)\ dx = \frac{e^{-itk^2/2}}{\sqrt{2\pi}}$$

where we compute $\widehat{\delta}(k)$ from first principles.

Now take the inverse Fourier transform:

$$\begin{aligned} (e^{-itH}\delta)(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}e^{-itk^2/2}\ dk \\ &= \frac{1}{2\pi}\int \exp\left[-\left(\frac{i}{2}\left(tk^2 - 2xk + \frac{x^2}{t}\right) - \frac{i}{2}\frac{x^2}{t}\right)\right]\ dk \\ &= \frac{e^{ix^2/2t}}{2\pi}\int \exp\left[\frac{-i}{2}\left(\sqrt{t}k - \frac{x}{\sqrt{t}}\right)^2\right]\ dk \\ &= \frac{e^{ix^2/2t}}{2\pi\sqrt{t}}\int \exp\left(\frac{-i}{2}u^2\right)\ du \end{aligned}$$

where we use the u-substitution $u = \sqrt{t}k - \frac{x}{\sqrt{t}}$ in the last step - and the last integral is just a number.

So just as desired, $(e^{-itH}\delta)(x) = K(t,x)$, where $K(t,x) = \dfrac{e^{\frac{i}{2}\frac{x^2}{t}}}{c(t)}$, and the normalizing factor is

$$\frac{1}{c(t)} = \frac{1}{2\pi}\frac{1}{\sqrt{t}}\int_{\mathbb{R}} \exp\left(\frac{-i}{2}u^2\right)\ du$$

Note that this integral is not always absolutely convergent (this is similar, for instance, to the fact that a series does not converge if its summand terms do not go to zero). However, to evaluate it, we can consider $\lim_{R\to\infty}\int_{-R}^{R}$, or alternatively, consider

$$\int e^{-\tau u^2/2}\ du$$

where $\tau \in \mathbb{C}$ is close to i, but has a small positive real part.

This then converges absolutely; we then take the limit as $\tau \to i$. Let's do it:

$$\begin{aligned}\int_{\mathbb{R}} e^{-\tau u^2/2}\ du &= \sqrt{\int_{\mathbb{R}} e^{-\tau x^2/2}\ dx \cdot \int_{\mathbb{R}} e^{-\tau y^2/2}\ dy} \\ &= \sqrt{\iint_{\mathbb{R}^2} e^{-\tau(x^2+y^2)/2}\ dx\ dy}\end{aligned}$$

Now convert to polar coordinates, and the Jacobian is r (this procedure is very well-known), and compute

$$= \sqrt{\int_0^\infty \int_0^{2\pi} e^{-\tau r^2/2} r\ d\theta\ dr} = (v = \frac{r^2}{2}) = \sqrt{2\pi \int_0^\infty e^{-\tau v}\ dv} = \sqrt{\frac{2\pi}{\tau}}$$

We now take the limit as $\tau \to i$. Thus,

$$K(t,x) = \frac{e^{ix^2/2t}}{2\pi\sqrt{t}} \cdot \sqrt{\frac{2\pi}{i}} = \frac{e^{ix^2/2t}}{\sqrt{2\pi i t}}$$

This is how to do the "simplest path-integral in the world"!

Next time: We throw in a potential - then it may not be exactly solvable, but we can still go some of the distance.

8. Feb 27, 2007: More examples of path integrals

8.1. **A potential problem.** Now let's consider a quantum particle of mass m on the real line, but in a potential

$$V : \mathbb{R} \to \mathbb{R}$$

Let's compute its time evolution using a path integral, using the fact that we already did (this for) the free particle. Our particle can trace out any path

$$\gamma : [0,T] \to \mathbb{R}$$

(where now, without loss of generality, we start the clock ticking at 0), and its action is

$$S(\gamma) = \int_0^T \left(\frac{m}{2}\dot{\gamma}(t)^2 - V(\gamma(t))\right) dt$$

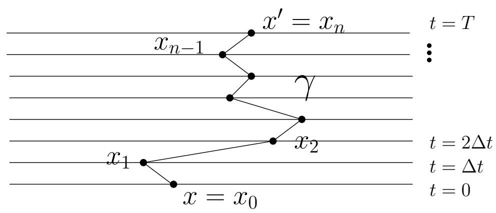
so the path integral philosophy tells us:

Given a wavefunction $\psi_0 \in L^2(\mathbb{R})$ at time 0, it will evolve to $\psi_T \in L^2(\mathbb{R})$ at time T, where

$$\psi_T(x') = \int_{x\in\mathbb{R}} \int_{\gamma \in P_{x\to x'}} e^{iS(\gamma)/\hbar}\psi_0(x)\ \mathscr{D}\gamma\ dx$$

(Here, $P_{x\to x'} = \{\gamma : [0,T] \to \mathbb{R}, \gamma \text{ piecewise regular}, \gamma(0) = x, \gamma(T) = x'\}$.)

To do this, we first integrate over piecewise linear paths like



(Here, $\Delta t = T/n$, and we only consider *regular* partitions of $[0,T]$. We now set $x_k = \gamma(k\Delta t)$, $\Delta x_k = x_k - x_{k-1}$ whenever defined.)

We now take the limit $n \to \infty$, if possible. So we hope

$$\psi_T(x') = \lim_{n\to\infty} \int_{\mathbb{R}^n} e^{iS(\gamma)/\hbar}\psi_0(x_0)\frac{dx_0}{c(\Delta t)} \cdots \frac{dx_{n-1}}{c(\Delta t)}$$

where

$$S(\gamma) = \int_0^T \left(\frac{m}{2}\dot{\gamma}(t)^2 - V(\gamma(t))\right) dt \sim \sum_{k=1}^n \left(\frac{m}{2}\left(\frac{\Delta x_k}{\Delta t}\right)^2 \Delta t - V(x_{k-1})\right) \Delta t$$

where the last expression merely an approximation, not an equality. Thus, we have decided to approximate $\displaystyle\int_{(k-1)\Delta t}^{k\Delta t} V(\gamma(t))$ by $V(x_{k-1})$. (Note that as $n \to \infty$, this approximation might not matter!)

So, we hope:

$$\psi_T(x') = \lim_{n\to\infty} \int_{\mathbb{R}^n} \prod_{k=1}^{n} e^{\frac{im}{2\hbar}\frac{(\Delta x_k)^2}{\Delta t}} e^{\frac{-i}{\hbar}V(x_{k-1})\Delta t} \cdot \psi_0(x_0) \frac{dx_0}{c(\Delta t)} \cdots \frac{dx_{n-1}}{c(\Delta t)}$$

Hence we start with our wavefunction ψ_0, and repeatedly applying the following two types of operators in alternation:

(1) multiplication by $\exp\left(\frac{-i}{\hbar} V \Delta t\right)$, and
(2) evolving it for a time Δt as if it were a free particle.

The latter step, we've seen, amounts to the operator

$$\psi \mapsto e^{-iH\Delta t/\hbar}\psi$$

where H is the *Hamiltonian* for a free particle:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

So $\psi_T = \lim_{n\to\infty} \left(e^{-iH\Delta t/\hbar} e^{-iV\Delta t/\hbar}\right)^n \psi_0$.

8.2. **The Lie-Trotter Theorem and self-adjoint operators.** We now need the following theorem.

Theorem 8.1 (Lie-Trotter)**.** *If A and B are (possibly unbounded) self-adjoint operators defined on a Hilbert space $\mathscr{H}$ (with dense domains $D(A)$, $D(B)$ respectively), so that $A+B$ is essentially self-adjoint on $D(A)\cap D(B)$, then*

$$e^{i(A+B)t}\psi = \lim_{n\to\infty} (e^{iAt/n} e^{iBt/n})^n \psi$$

for all $\psi \in \mathscr{H}$.

Remark 8.2.

(1) Unbounded self-adjoint operators are only "densely defined" on $\mathscr{H}$. Thus, $A+B$ is only necessarily defined on $D(A)\cap D(B)$.
(2) Now, if A is such an operator, then for each t, e^{iAt} is unitary. We can thus define $e^{iAt/n}e^{iBt/n}$ on $D(A)\cap D(B)$, and the definition can be extended (for *this* operator) to all of $\mathscr{H}$.
(3) For details, see Volume 1, *Methods in Modern Mathematical Physics*, by Reed and Simon.

In fact, H and V (i.e. mult_V) are self-adjoint on $L^2(\mathbb{R})$, and $H+V$ is essentially self-adjoint on $D(H)\cap D(V)$, if V is reasonably nice - e.g. continuous and bounded below. So in this case,

$$\psi_T = \lim_{n\to\infty} \left(e^{-iH\Delta t/\hbar} e^{-iV\Delta t/\hbar}\right)^n \psi_0$$

exists and equals $e^{-i(H+V)T/\hbar}\psi_0$. So we get that $\psi_T = e^{-i(H+V)T/\hbar}\psi_0$, and if $\psi_0 \in D(H+V)$, we can differentiate this and get (set $T \leftrightarrow t$):

$$\frac{d}{dt}\psi_t = \frac{-i}{\hbar}(H+V)\psi_t$$

which is *Schrödinger's equation.*

8.3. **Generalization to complete Riemannian manifolds.** We can also handle the case of a particle on a complete (connected) Riemannian manifold Q (see the lecture on January 16, 2007). Here again,

$$H = -\frac{\hbar^2}{2m}\nabla^2$$

and "V" $=$ mult$_V$ ($V : Q \to \mathbb{R}$) are self-adjoint operators on $L^2(Q)$, and if V is continuous and bounded below, $H+V$ is essentially self-adjoint on $D(H) \cap D(V)$.

Remark 8.3.

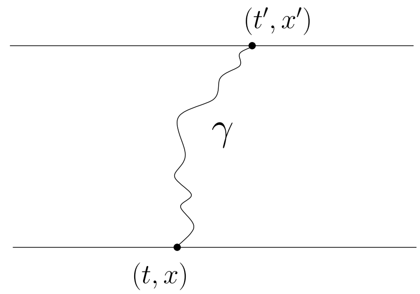
(1) The notion of piecewise linear paths does not make sense here. But we can still go for "piecewise geodesic" paths, each piece in a small enough coordinate-patch.
(2) Then, $e^{-iHT/\hbar}$ is not exactly an integral over such paths with only a few pieces; we need to take the limit as the number n of pieces goes to infinity.
(3) So again, the final answer is as desired, but only in the limit - so the intermediate steps are only approximations now, unlike the case $Q = \mathbb{R}$.
(4) Some people study this after applying Wick rotation; this leads to the study of the *Heat equation* and of Brownian motion on manifolds.

Upshot. So again, skipping lots of steps, we obtain this formula for the time evolution of a wavefunction for a particle on Q with potential V:

$$\psi_T = \lim_{n\to\infty}\left(e^{-iHT/n\hbar}e^{-iVT/n\hbar}\right)^n \psi_0 = e^{-i(H+V)T/\hbar}\psi_0$$

where $H+V \leftrightarrow -\dfrac{\hbar^2}{2m}\nabla^2 +$ mult$_V$. (See the notes from January 16, 2007!) Here, the operator ∇^2 is defined for general Q, and $H+V$ is the *Hamiltonian* for our particle.

8.4. **Back to the general picture.** Now let's return to our general story. We have a category $\mathcal{C}$ whose objects are "configurations" and whose morphisms are "paths". In the example we just saw, objects were points in $\mathbb{R}\times Q$ (spacetime) and a morphism $\gamma : (t,x) \to (t',x')$ is a path $\gamma : [t,t'] \to Q$ so that $\gamma(t) = x, \gamma(t') = x'$.



We also have a functor $S : \mathcal{C} \to \mathbb{R}$ (where $\mathbb{R}$ is the category with one object, reals as morphisms, $+$ as composition) serving as our action, giving

$$e^{iS/\hbar} : \mathcal{C} \to U(1) \subset \mathbb{C}$$

Question. How do we get a Hilbert space from this general framework? In the examples we just saw, we could use $L^2(Q)$ - but there's no "Q" in general! Q came from our ability to "slice" he set of objects $\mathbb{R} \times Q$ into slices $\{t = \text{constant}\}$, called *Cauchy surfaces* - surfaces on which we can freely specify "initial data" (= *Cauchy data*) for our wavefunction (and then we can solve differential equations given such Cauchy data - for example, Schrödinger's equation).

8.5. **Digression of the day: Cauchy surfaces.** In Newtonian mechanics, spacetime is $\mathbb{R} \times \mathbb{R}^3$, so Cauchy surfaces are the level sets for the first coordinate.

[figure: parallel family of horizontal straight lines]

In special relativity, spacetime is $\mathbb{R}^4$ with the Minkowski metric. So Cauchy surfaces can be one of many different families of parallel lines. *Lorentz transformations* take one family of Cauchy surfaces to another.

[figure: parallel families of horizontal straight lines, cutting one another transversely]

Finally, in general relativity, spacetime is a 4-manifold with a Lorentzian metric; then Cauchy surfaces can be very badly behaved. They may not even exist! For instance, consider $S^1 \times \mathbb{R}^3$. There are closed timelike loops here.

(In fact, the problem in many (most?) time-traveller paradoxes in science fiction, is that of a lack of Cauchy surfaces. You *cannot* be in two places at once in the same universe!)

9. Mar 6, 2007: Hilbert spaces and operator algebras from categories

Suppose we have a category $\mathcal{C}$ of "configurations and processes" and an "action" functor $S : \mathcal{C} \to (\mathbb{R}, +)$ giving the phase (set $\hbar = 1$ henceforth) $e^{iS} : \mathcal{C} \to (U(1), \cdot)$ describing the amplitude for any process to occur.

How do we get a Hilbert space out of this? (One approach is to try and get a Cauchy surface; however, we will try something different.) Here's one avenue of attack:

9.1. (Pre-)Hilbert spaces from categories.

First, as a zeroth approximation to our Hilbert space, form a vector space as follows: let $\mathrm{Ob}(\mathcal{C})$ (resp. $\mathrm{Mor}(\mathcal{C})$) be the set of all objecs (resp. morphisms) in $\mathcal{C}$. Then we have $s, t : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$, assigning to any morphism $\gamma : x \to y$ its *source* $s(\gamma) = x$ and *target* $t(\gamma) = y$ respectively.

Form the vector space $\mathrm{Fun}(\mathrm{Ob}(\mathcal{C}))$ of "nice" complex-valued functions on $\mathrm{Ob}(\mathcal{C})$ - where we'll have to see what "niceness" is required.

Then define for $\psi, \phi \in \mathrm{Fun}(\mathrm{Ob}(\mathcal{C}))$ an "inner product":

$$\langle \phi, \psi \rangle := \int_{\gamma : x \to y} e^{iS(\gamma)} \overline{\phi(y)} \psi(x) \; \mathscr{D}\gamma \mathscr{D}x \mathscr{D}y = \int_{\mathrm{Mor}(\mathcal{C})} e^{iS(\gamma)} \overline{\phi(t(\gamma))} \psi(s(\gamma)) \; \mathscr{D}\gamma$$

Remark 9.1.

(1) For this to make sense, we need a measure on $\mathrm{Mor}(\mathcal{C})$, and ψ, ϕ should be nice enough so that the integral converges - for instance, $\psi \circ s, \phi \circ t \in L^2(\mathrm{Mor}(\mathcal{C}))$.

(2) Under "nice" conditions, given a measure $\mathscr{D}\gamma$ on $\mathrm{Mor}(\mathcal{C})$, we can find a measure on $\mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C})$, and for any point (x, y) here, a measure on $\mathrm{Mor}_{\mathcal{C}}(x, y)$, so that $\mathscr{D}\gamma \leftrightarrow \mathscr{D}\gamma \; \mathscr{D}x \; \mathscr{D}y$.

Now we have questions:

(1) Is $\langle \phi, \psi \rangle$ linear in ψ and antilinear in ϕ?
(2) Is $\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle$?
(3) Is $\langle -, - \rangle$ nondegenerate? That is, given ϕ so that $\langle \phi, \psi \rangle = 0 \; \forall \psi$, is $\phi = 0$?
(4) Is $\langle \psi, \psi \rangle \geq 0$ for all ψ?

Consider these in turn:

(1) is obvious if the integral is well-behaved.

(2) is more interesting:
  (a) On the one hand,

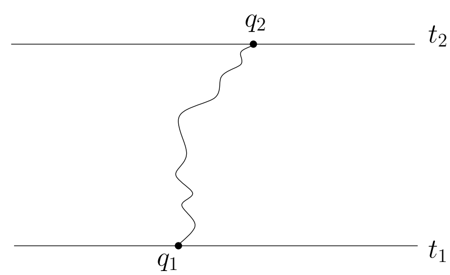
$$\overline{\langle \phi, \psi \rangle} = \int_{\gamma : x \to y} e^{-iS(\gamma)} \overline{\psi(x)} \phi(y) \; \mathscr{D}y \; \mathscr{D}x \; \mathscr{D}y$$

whereas

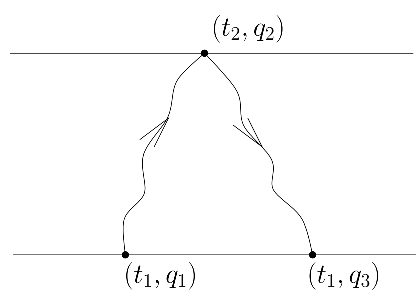
$$\langle \psi, \phi \rangle = \int_{\gamma : x \to y} e^{iS(\gamma)} \overline{\psi(x)} \phi(y) \; \mathscr{D}\gamma \mathscr{D}x \mathscr{D}y = \int_{\gamma : y \to x} e^{iS(\gamma)} \overline{\psi(y)} \phi(x) \; \mathscr{D}\gamma \mathscr{D}x \mathscr{D}y$$

upon relabelling $x \leftrightarrow y$. Thus, the equality of the two comes from "time reversal symmetry". It's easy if $\mathcal{C}$ is a groupoid, since then, given $\gamma : x \to y$, we get $\gamma^{-1} : y \to x$ - and since S is a functor, $S(\gamma^{-1}) = -S(\gamma)$. So we'll get $\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle$ if the measure $\mathscr{D}\gamma$ on $\mathrm{Mor}(\mathcal{C})$ is preserved by the transformation $^{-1} : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$.

(b) But - our favourite example is not a groupoid! Recall - given a manifold Q, we have a category with $\mathrm{Ob}(\mathcal{C}) = \mathbb{R} \times Q$, and a morphism $\gamma : (t_1, q_1) \to (t_2, q_2)$ is a path $\gamma : [t_1, t_2] \to Q$ with $\gamma(t_i) = q_i$.

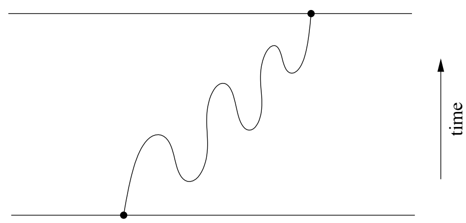


Here, we've been assuming $t_1 \leq t_2$, so this is not a groupoid. We *could* adjoin inverses to get a groupoid, but then we'd get morphisms like



Such morphismsm do indeed show up in Feynman diagrams involving antimatter, but would require further thoughts.

(c) Research topics:
   (i) Study Feynman's original work on path integrals for a special-relativistic particle, and see if he allowed paths like



[figure: path that has parts moving both ways in time]

   (ii) If so, formalize what he did using some category $\mathcal{C}$. Is it a groupoid, or merely a $*$-*category*?

(d) A $*$-*category* is a category $\mathcal{C}$ with a contravariant functor $* : \mathcal{C} \to \mathcal{C}$ that is the identity on objects, and satisfies $** = 1_{\mathcal{C}}$. Equivalently, for any morphism $\gamma : x \to y$, there is a morphism $\gamma^* : y \to x$ so that
   (i) $(\gamma_1 \circ \gamma_2)^* = \gamma_2^* \circ \gamma_1^*$
   (ii) $(\gamma^*)^* = \gamma$

(These imply that $(1_x)^* = 1_x \; \forall x \in \mathrm{Ob}(\mathcal{C})$.) This is also called a *category with involution*, an *involutive category*, or in quantum computing, a †-*category*.

(e) We now make some remarks.

Remark 9.2.
   (i) An obvious eample is a groupoid with one object - also called a group! Then $*$ is the inverse map on morphisms.
   (ii) The main example, however, is the category of Hilbert spaces and bounded linear operators, denoted by Hilb: given $T : H \to H'$, we get the *adjoint operator* $T^* : H' \to H$, defined by

$$\langle T^*\phi, \psi\rangle := \langle \phi, T\psi\rangle$$

   (iii) The requirement that $x^* = x$ for all objects x of $\mathcal{C}$ is somewhat "evil"; we might ask for a "non-strict" (ala monoidal categories) version only.

(3) $\langle -, -\rangle$ is usually degenerate, but that's not bad; we can form the vector subspace $K \subset \mathrm{Fun}(\mathrm{Ob}(\mathcal{C}))$ by

$$K = \{\psi : \langle \phi, \psi\rangle = 0 \; \forall \phi \in \mathrm{Fun}(\mathrm{Ob}(\mathcal{C}))\}$$

and form the quotient space

$$H_0 = \mathrm{Fun}(\mathrm{Ob}(\mathcal{C}))/K$$

on which we have $\langle -, - \rangle$ defined by

$$\langle [\phi], [\psi] \rangle := \langle \phi, \psi \rangle$$

Then this new $\langle -, - \rangle$ is nondegenerate on H_0.

(4) Is $\langle \psi, \psi \rangle \geq 0$?

To get this, we need some extra conditions - but we'd need to look at some examples to find *nice* sufficient conditions.
This is somehow related to "reflection positivity" in the Osterwalder-Schrader Theorem.

Anyhow, if we get that these properties all hold, then H_0 is called a *pre-Hilbert space*; we can then *complete* it to get a Hilbert space.

9.2. **Operators and multiplying them.** Besides the issue of producing a Hilbert spaces, there's the issue of operators. How can we get some "nice" operators in $\mathrm{Fun}(\mathrm{Ob}(\mathcal{C}))$?

We can get them from elements $F \in \mathrm{Fun}(\mathrm{Mor}(\mathcal{C}))$, some space of "nice" complex-valued functions on $\mathrm{Mor}(\mathcal{C})$:

$$(F\psi)(y) := \int_{\gamma : x \to y} F(\gamma)\psi(x) \ \mathscr{D}\gamma \ \mathscr{D}x$$

where "nice" means this converges.

In fact, we get an algebra of such operators with some luck:

$$GF(\gamma) = \int_{\mathscr{P}} G(\gamma_2)F(\gamma_1) \ \mathscr{D}p$$

where we integrate over the set

$$\mathscr{P} := \{(\gamma_1, \gamma_2) \in \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C}) : \gamma_2 \circ \gamma_1 = \gamma\}$$

and $\mathscr{D}p$ is a measure on $\mathscr{P}$.

Remark 9.3.

(1) This is "convolution"; $\mathrm{Fun}(\mathrm{Mor}(\mathcal{C}))$ is called the "category algebra" of $\mathcal{C}$.
(2) If we're working over a *groupoid* $\mathcal{C}$, the above integral can be converted to an integral only over morphisms to one point, by a "change of variables"; moreover, the measure here is one that has already shown up earlier above.

10. Mar 13, 2007: The big picture

10.1. **The case of finite categories.** Last time, we sketched how to get a Hilbert space from a category $\mathcal{C}$ (of "configurations" and "processes") equipped with an "amplitude" functor

$$A : \mathcal{C} \to U(1)$$

There are lots of subtleties involving analysis, but these evaporate when $\mathcal{C}$ is finite (so all morphism spaces are finite sets, all integrals are finite sums). Then we form the vector space $\mathrm{Fun}(\mathrm{Ob}(\mathcal{C}))$ - which now means *all* functions $\psi : \mathrm{Ob}(\mathcal{C}) \to \mathbb{C}$.

$\mathrm{Fun}(\mathrm{Ob}(\mathcal{C}))$ is isomorphic to $\mathbb{C}[\mathrm{Ob}(\mathcal{C})]$ - the space of formal linear combinations of objects of $\mathbb{C}$. (This corresponds to allowing *superpositions* in quantum mechanics.)

Then we define a $\mathbb{C}$-sesquilinear map

$$\langle -, - \rangle : \mathbb{C}[\mathrm{Ob}(\mathcal{C})] \times \mathbb{C}[\mathrm{Ob}(\mathcal{C})] \to \mathbb{C}$$

by

$$\langle y, x \rangle := \sum_{\gamma : x \to y} A(\gamma) \ \forall x, y \in \mathrm{Ob}(\mathcal{C})$$

Recall that $A = e^{iS/\hbar}$ here. (The prefix "sesqui" means "one-and-a-half", and this is particularly appropriate, since it is antilinear - hence only $\mathbb{R}$-linear - in the first coordinate, but $\mathbb{C}$-linear in the second coordinate.)

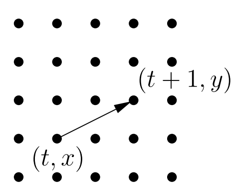
We're doing a "path integral", but now it's a sum over morphisms - we're implicitly using counting measure on $\mathrm{Hom}_{\mathcal{C}}(x, y)$.

Take $\mathbb{C}[\mathrm{Ob}(\mathcal{C})]$ and mod out by

$$\{\psi \in \mathbb{C}[\mathrm{Ob}(\mathcal{C})] : \langle \psi, \phi \rangle = 0 \ \forall \phi \in \mathbb{C}[\mathrm{Ob}(\mathcal{C})]\}$$

to get a vector space H with a nondegenerate sesquilinear form on it; if that's positive definite, then H is a (finite-dimensional) Hilbert space.

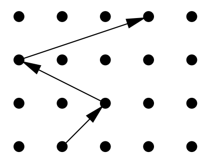
10.2. **Example: particle on a line.** Alex Hoffnung and I (=JB) have been looking at examples like this:



In this example,

- $\mathrm{Ob}(\mathcal{C}) = \{1, 2, \ldots, T\} \times \{1, 2, \ldots, X\}$.
- Morphisms in $\mathcal{C}$ are freely generated (under the composition / concatenation operation, and including the identity) by morphisms $\gamma : (t, x) \to (t+1, y)$ for all $t \in \{1, 2, \ldots, T-1\}$ and $x, y \in \{1, 2, \ldots, X\}$.

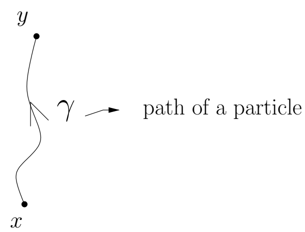
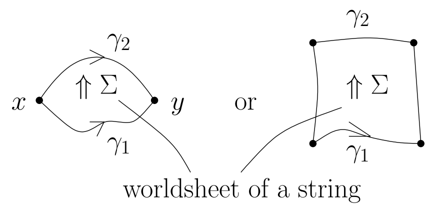
So a typical morphism in $\mathcal{C}$ is like



Now $\mathcal{C}$ is a "quiver".

If you choose the amplitude $A : \mathcal{C} \to U(1)$ to be a discretized version of the amplitude for a particle on a line, we recover standard physics in the continuous limit.

10.3. **From particles to strings.** We really want to categorify all this. We now make a table as we have done earlier; we shall make explanatory remarks later on.

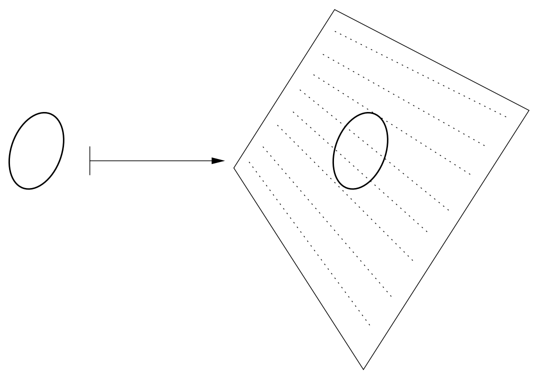
<table>
<tr><th>Particles</th><th>Strings</th></tr>
<tr><td>(1) A category $\mathcal{C}$.

</td><td>(1) A 2-category (or double category) $\mathcal{C}$.

</td></tr>
<tr><td>(2) A functor $A : \mathcal{C} \to U(1) \subset \mathbb{C}$. Here, $U(1)$ is the 1-category with one object $*$, and the morphisms are the elements of $U(1)$.</td><td>(2) A 2-functor $: \mathcal{C} \to U(1)[1]$. (This is explained later.)</td></tr>
<tr><td>(3) From $A : \mathcal{C} \to U(1)$, we try to build a Hilbert space - but first we form the vector space $\mathrm{Fun}(\mathrm{Ob}(\mathcal{C}))$, which, if $\mathcal{C}$ is finite, is just
$\mathrm{Hom}(\mathrm{Ob}(\mathcal{C}), \mathbb{C}) \cong \mathbb{C}[\mathrm{Ob}(\mathcal{C})]$</td><td>(3) From $A : \mathcal{C} \to U(1)[1] - \mathrm{Tor} \cong U(1)[1]$, we try to build a 2-Hilbert space $FUN(OB(\mathcal{C}))$, which, if $\mathcal{C}$ is finite, is just
$\mathrm{Hom}(OB(\mathcal{C}), \mathrm{Vect}_{\mathbb{C}}) \cong$ (we hope) $\mathrm{Vect}_{\mathbb{C}}[OB(\mathcal{C})]$.

Here, $OB(\mathcal{C})$ could be the category formed by discarding the 2-morphisms in our 2-category $\mathcal{C}$ - but this only works if $\mathcal{C}$ is strict. What to do in general? Good questions.</td></tr>
<tr><td>(4) We define $\langle -, - \rangle$ on $\mathbb{C}[\mathrm{Ob}(\mathcal{C})]$ by
$\langle y, x \rangle = \sum_{\gamma : x \to y} A(\gamma)$.
Here, we use $U(1) \hookrightarrow \mathbb{C}$ to add elements of $U(1)$ and get elements of $\mathbb{C}$.</td><td>(4) $\langle -, - \rangle$ on $\mathrm{Vect}_{\mathbb{C}}$ should satisfy
$\langle y, x \rangle = \bigoplus_{\gamma : x \to y} A(\gamma) \in \mathrm{Vect}_{\mathbb{C}}$.
Here we use $U(1) - \mathrm{Tor} \hookrightarrow \mathrm{Vect}_{\mathbb{C}}$; this is explained below.</td></tr>
</table>

Remark 10.1. We present some facts about torsors below; let us first address the other points in the table above.

(1) For any abelian group A and $n \geq 0$, we can form an n-category $A[n]$: the objects form a singleton set $\{*\}$, the 1-morphisms are $\{1_*\}$, the 2-morphisms are $\{1_{1_*}\}$, and so on, until the n-morphisms (between the unique $(n-1)$-morphism and itself). This set is different - and equals A.

(2) While defining $\langle -,- \rangle$ in the case of strings, we use that $U(1) - \mathrm{Tor} \hookrightarrow \mathrm{Vect}_{\mathbb{C}}$, sending a torsor to its corresponding 1-dimensional vector space (that contains the circle $U(1)$). (In fact, $U(1) - \mathrm{Tor} \hookrightarrow \mathbb{C} - \mathrm{Tor}$?)



Moreover, this vector space is actually a *Hilbert* space. For more on this, see Daniel Freed's *Higher algebraic structures and quantization*.

10.4. **Digression on torsors.** Given any group G, a *G-torsor* is a G-set isomorphic to G (viewed as a G-set via left-multiplication). Denote the set of G-torsors by $G - \mathrm{Tor}$. In other words, "a torsor is a group that has forgotten its identity".

Thus, G-torsor morphisms are G-set maps that are also bijections.

(1) If G is abelian, then $G - \mathrm{Tor}$ is a monoidal category with
$$X \otimes Y := X \times Y / \{(xg = gx, y) \sim (x, gy)\}$$
where $X, Y \in G - \mathrm{Tor}$ and $g \in G$ acts on the right on X (since G is abelian and acts on X).

(2) If G is abelian, then $G - \mathrm{Tor}$ is a 2-category as follows: it has
one object $*$;
G-torsors as morphisms, composed using $\otimes$ defined above; and
G-torsor morphisms as 2-morphisms.

(3) Next, recall the definition of the 2-category $G[1]$ for the abelian group G. We now have that
$G[1]$ *is a skeleton of* $G - \mathrm{Tor}$ - so we have $G[1] \cong G - \mathrm{Tor}$.

To see this, we need one "special" morphism for every object. Thus, identify one of the torsors with G! This is to correspond to the identity morphism, since we can verify that $G \otimes G = G$ (where $\otimes$ was defined above).