More Examples of 2-Categories

2) \textbf{Cat} - the 2-category with

- categories as objects
- functors as morphisms
- natural transformations as 2-morphisms

In particular, we can compose natural transformations both horizontally and vertically.

\textbf{Vertical composition:}

We need \( \alpha \beta : F \Rightarrow H \), and in particular given \( c \in C \)

\[ (\alpha \beta)_c : F_c \rightarrow H_c \]

We take the composite

\[ F_c \xrightarrow{x_c} G_c \xrightarrow{\beta_c} H_c \]

to be \( (\alpha \beta)_c \). Check that \( \alpha \beta \) is natural.
Horizontal composition:

we need $\alpha \circ \beta : F \circ G \Rightarrow F' \circ G'$,
and in particular given $c \in C$

$(\alpha \circ \beta)_c : (F \circ G)_c \longrightarrow (F' \circ G')_c$

$G(F_c) \quad G'(F_c)$

$\alpha$ gives us

$\alpha_c : F_c \longrightarrow F'_c$

and the functors $G$ & $G'$ map this to

$G(F_c) \xrightarrow{G(\alpha_c)} G(F'_c)$

$G'(F_c) \xrightarrow{G'(\alpha_c)} G'(F'_c)$

and $\beta$ gives

$G(F_c) \xrightarrow{G(\beta_c)} G(F'_c)$

$C(F_c) \xrightarrow{G'(\alpha_c)} G'(F'_c)$

which commutes by naturality of $\beta$. So we can use either composite to define

$(\alpha \circ \beta)_c : (F \circ G)(c) \longrightarrow (F' \circ G')(c)$

Check that $\alpha \circ \beta$ is natural.
Homework: Check that $\text{Cat}$ is a 2-category — a strict 2-category. So check:

- $\alpha \beta$ is natural
- $\alpha \circ \beta$ is natural
- Associativity & r/l unit laws for vertical and horizontal composition
- Interchange law

3) For any topological space $X$, there's a 2-category $\Pi_2(X)$ with

- Points of $X$ as objects
- Paths in $X$ as morphisms
- Homotopy classes of path homotopies as 2-morphisms

Unlike $\text{Cat}$, $\Pi_2(X)$ is a weak 2-category: given paths
we don't have \((\alpha\beta)\gamma = \alpha(\beta\gamma)\):

\[
\begin{array}{c}
\alpha \rightarrow \beta \\
\beta \rightarrow \gamma
\end{array}
\xrightarrow{(\alpha\beta)\gamma} X
\]

\[
\begin{array}{c}
\alpha \\
\beta
\end{array}
\xrightarrow{\alpha(\beta\gamma)} X
\]

but we do have an associator: a 2-isomorphism from one to the other:

\[
\begin{array}{c}
\alpha \beta \\
\beta \gamma
\end{array}
\rightarrow X
\]

Similarly for \(l/r\) units:

\[
\begin{array}{c}
\alpha
\end{array}
\rightarrow X
\]

\[
\begin{array}{c}
\beta
\end{array}
\rightarrow X
\]

Need to check the pentagon and triangle identities.
In $T_2(X)$, every 2-morphism $h: \alpha \Rightarrow \beta$ has an inverse $h^{-1}: \beta \Rightarrow \alpha$: $hh^{-1} = 1$ & $h^{-1}h = 1$, so they're all 2-isomorphisms. Also, every morphism $f: x \rightarrow y$ has a weak inverse, i.e. $\tilde{f}: y \rightarrow x$ s.t. there exist 2-isomorphisms $h: \tilde{f}f \Rightarrow 1$ & $h': \tilde{f}f \Rightarrow 1$, so every 1-morphism is an equivalence.

(In Cat, a 2-iso. is called a natural isomorphism & an equivalence is called an equivalence.)

A 2-category where every morphism is an equivalence & every 2-morphism is a 2-isomorphism is a 2-groupoid, & $T_2(X)$ is the fundamental 2-groupoid of $X$.

4) There's a 2-category Top$_2$ with

- topological spaces as objects
- continuous maps as morphisms
- homotopy classes of homotopies between maps as 2-morphisms

This is strict.
5) There's a weak 2-category \( n\text{Cob}_2 \) for any \( n \geq 2 \) with (roughly)

- \((n-2)\)-dimensional manifolds as objects
- \((n-1)\)-dimensional manifolds with boundary (cobordisms) as morphisms
- \(n\)-dimensional manifolds with corners ("cobordisms between cobordisms") as 2-morphisms.

Jeffrey Morton has constructed this in his paper "A Double Bicategory of Cobordisms with Corners." In physics, the 2-morphisms here represent choices of spacetime, 1-morphisms represent choices of space, 0-morphisms represent manifolds that could be the boundary of space.
If \( n = 3 \), these boundaries (unions of circles) act like particles which can interact:

![Diagram](image)

In this framework, a (once) extended topological nice quantum field theory is a 2-functor

\[
Z : n\text{Cob}_2 \longrightarrow 2\text{Hilb}
\]

just as an ordinary TQFT is a nice functor

\[
Z : n\text{Cob} \longrightarrow \text{Hilb}
\]

where \( n\text{Cob} \) is a mere 1-category with

- \((n-1)\)-dim manifolds as objects
- \( n \)-dim cobordisms as morphisms

\((n=2)\)
6) In string theory we need a 2-category $\text{2Cob}_2^C$ which is like $\text{2Cob}_2$ but where 2-morphisms (2-manifolds w. corners) have a complex analytic structure. This is challenging to define.

7) A monoidal category is the same as a 2-category with one object, just as a monoid is the same as a category with one object:

Given a 2-category with just one object, we do a relabelling game:

2 morphisms $\longrightarrow$ morphisms
morphisms $\longrightarrow$ objects
one object $\longrightarrow$ ——— (forget about it!)

———
2-category w. 1 object

———
Category where we can "tensor" (compose) objects and "tensor" (horizontally compose) morphisms
E.g. the monoidal category \((\text{Vect}, \otimes)\) can be considered as a (weak) 2-category & interchange law says

\[(\alpha \otimes \beta)(\alpha' \otimes \beta') = (\alpha \alpha')(\beta \beta')\]

For any field \(k\), this \((\text{Vect}, \otimes)\) will base field \(k\) sits inside a weak 2-category \(\text{Bim}_k\) as:

- rings as objects
- \((R, R')\)-bimodules as morphisms from \(R\) to \(R'\)
- \((R, R')\)-bimodule homomorphisms as 2-morphisms

(\text{So, in the monoidal category } (\text{Vect}, \otimes), \text{ considered as a 2-category with one object, it is nice to think of the object as the base field})