AN EXAMPLE OF PATH INTEGRAL QUANTIZATION

We have a strategy for quantization given any category \( C \) (of "configurations" & "paths") and functor

\[ S : C \rightarrow (\mathbb{R}, +) \quad \text{(the "action")} \]

This gives a functor

\[ e^{iS/\hbar} : C \rightarrow (C, \cdot) \]

& we compute the "transition amplitude" from any object \( x \in C \) to any \( y \in C \) via:

\[ Z_h(x, y) = \int_{\gamma : x \rightarrow y} e^{iS(\gamma)/\hbar} d\gamma \]

Let's do an example — the free particle on \( \mathbb{R} \).
Here the objects of \( C \) form the set \( \mathbb{R}^2 \equiv (t, q) \)
& morphisms \( \gamma : (t, x) \rightarrow (t', q') \) are paths:

\[ \gamma : [t, t'] \rightarrow \mathbb{R} \]

s.t. \( \gamma(t) = q \) & \( \gamma(t') = q' \)
So: \[ Z_h^k(t, q, t', q') = \int_{\gamma: (t, q) \rightarrow (t', q')} e^{iS(\gamma)/\hbar} d\gamma \]

where \( S \) is the action for a free particle of mass \( m \):
\[
S(\gamma) = \int_t^{t'} L(\gamma(s), \dot{\gamma}(s)) \, ds
\]
where the Lagrangian is just
\[
L(q, \dot{q}) = \frac{m}{2} \dot{q}^2
\]
since there's no potential.

To do the integral over all paths \( \gamma \), we first integrate only over piecewise linear paths like:

For some chosen times \( t = t_1 < t_2 < \ldots < t_{N+1} = t' \).
To integrate over all these piecewise-linear paths, we just integrate over $x_2, \ldots, x_N$ where $x_i = \gamma(t_i)$. Then we'll try to show that these integrals over piecewise-linear paths converge as the "mesh spacing" $\max \{t_{i+1} - t_i\}$ goes to zero. First, let's see what these integrals look like — let's compute one:

$$A_i = \int_{R^{N-1}} e^{\frac{i}{h} \int_0^t \frac{m}{2} \gamma(s)^2 \, ds} \, dx_2 \, dx_3 \ldots \, dx_N$$

But $\gamma$ is piecewise-linear on the $i$th piece $[t_i, t_{i+1}]$, we have

$$\dot{\gamma}(s) = \frac{x_{i+1} - x_i}{t_{i+1} - t_i} =: \frac{\Delta x_i}{\Delta t_i}$$

(where $\Delta x_i = x_{i+1} - x_i$, $\Delta t_i = t_{i+1} - t_i$) so we get

$$A = \int_{R^{N-1}} e^{\frac{i}{h} \frac{m}{2} \sum_{j=1}^N (\frac{\Delta x_j}{\Delta t_j})^2 \Delta t_j} \, dx_2 \ldots dx_N$$

or exactly

$$A = \int_{R^{N-1}} e^{\frac{i}{h} \frac{m}{2} \sum_{j=1}^N (\frac{\Delta x_j}{\Delta t_j})^2 \Delta t_j} \frac{dx_2}{c_2} \ldots \frac{dx_N}{c_N}$$

where we rescale Lebesgue measure by a normalizing factor $c_i$ on the $i$th piece which depends on $t_{i+1} - t_i$. These normalizing
factors are needed to get convergence as the mesh spacing goes to zero. But much better: if we pick the \(c_i\) correctly, \(A\) is actually \text{independent} of the mesh.

\[ t = t_1 < t_2 < t_3 < \ldots < t_N < t_{N+1} = t' \]

so convergence is trivial. In other words, we can compute \(Z_k((t,q),(t',q'))\) as an integral over linear paths:

\begin{align*}
q' & \quad t' = t_2 \\
q & \quad t = t_1
\end{align*}

of which there is just one! To prove that \(A\) is independent of the mesh, let's think instead about the rule for evolving a wavefunction \(\Psi\) in time:

\begin{align*}
\Psi(t',q') \in C \\
\Psi(t,q) \in C
\end{align*}
\[ \psi(t', q') = \int \int \int_{\mathbb{R}} e^{iS(q)/\hbar} \psi(t, q) \, dq \, df \]

\[ = \int_{\mathbb{R}} Z_k(t, q), (t', q') \psi(t, q) \, dq \]

When we approximate \( Z_k \) by integrating over piecewise linear paths, we get

\[ \tilde{\psi}(t', q') = \int_{\mathbb{R}^N} e^{i \Delta x_i / \Delta t_i} \cdots e^{i \Delta x_N / \Delta t_N} \psi(t, q) \, dx_1 / c_1 \cdots dx_N / c_N \]

\[ = \int_{\mathbb{R}^N} K(\Delta t_N, \Delta x_N) \cdots K(\Delta t_1, \Delta x_1) \psi(t, q) \, dq \, dx_1 \cdots dx_N \]

where

\[ K(\Delta t_i, \Delta x_i) = \frac{1}{c_i} e^{i \Delta x_i / \Delta t_i} \]

So to show \( \tilde{\psi}(t', q') \) is independent of the mesh, we just need to show \( \tilde{\psi}(t', q') \) is independent of the mesh for all \( \psi(t, q) \).
To show this, it's enough to show:

\[ K(t_3 - t_1, x_3 - x_1) = \int K(t_3 - t_2, x_3 - x_2) K(t_2 - t_1, x_2 - x_1) \, dx_2 \]

for a suitable choice of \( c_i \).