Hilbert Spaces & Operator Algebras
from Categories

Suppose we have a category $C$ of "configurations & processes" and an "action" functor

$$S : C \to (\mathbb{R}, +)$$

giving

$$e^{iS} : C \to (U(1), \cdot)$$

describing the amplitude for any process to occur. How do we get a Hilbert space out of this? Here's one avenue of attack: First, as a 0th approximation to our Hilbert space, form a vector space as follows. Let $\text{Ob}(C)$ be the set of all objects of $C$ & $\text{Mor}(C)$ be the set of all morphisms in $C$. We have the source and target maps:

$$s, t : \text{Mor}(C) \to \text{Ob}(C)$$

assigning to each morphism $\gamma : x \to y$ its source $s(\gamma) = x$ and target $s(\gamma) = y$. Form the vector space $\text{Fun}(\text{Ob}(C))$ of "nice" complex functions on $\text{Ob}(C)$, where we'll have to see what sort of niceness we need.
Then, define for $\psi, \phi \in \text{Fun}(\text{Ob}(C))$ an "inner product":

$$\langle \phi, \psi \rangle = \int_{x \rightarrow y} e^{iS(y)} \overline{\phi(y)} \psi(x) \, dy \, dx \, dy$$

$$= \int_{\text{Mor}(C)} e^{iS(y)} \overline{\phi(t(y))} \psi(s(y)) \, dx$$

For this to make sense we really need a measure on $\text{Mor}(C)$ & $\psi, \phi$ should be nice enough so that the integral converges — e.g. $\phi \circ t, \psi \circ s \in L^2(\text{Mor}(C))$

Now we have questions:

1) Is $\langle \phi, \psi \rangle$ linear in $\psi$ and conjugate linear in $\phi$?

2) Is $\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle$?

3) Is $\langle - , - \rangle$ nondegenerate? (Given $\phi$ s.t. $\langle \phi, \psi \rangle = 0$ for all $\psi$, is $\phi = 0$?)

4) Is $\langle \psi, \psi \rangle \geq 0$?
Consider each of these in turn

1) is obvious if the integral is well behaved.

2) is more interesting:

\[ \langle \phi, \psi \rangle = \langle \psi, \phi \rangle \]

\[ \int e^{-iS(\gamma)} \overline{\psi}(x) \phi(y) \delta \, dx \, dy \quad \int e^{iS(\gamma)} \overline{\psi}(x) \phi(y) \delta \, dx \, dy \]

This is related to time reversal symmetry. It is immediate if \( C \) is a groupoid, since then given

\[ \gamma : x \rightarrow y \]

we get

\[ \gamma^{-1} : y \rightarrow x \]

and since \( S \) is a functor

\[ S(\gamma^{-1}) = -S(\gamma) \]

So we'll get \( \langle \phi, \psi \rangle = \langle \psi, \phi \rangle \) if the measure \( dY \) on \( \text{Mor}(C) \) is preserved by the transformation.

\[ \gamma^{-1} : \text{Mor}(C) \rightarrow \text{Mor}(C) \]
But our favorite example is not a groupoid! Recall given a manifold $Q$ we have a category $\mathcal{C}$:

$$\text{Ob}(\mathcal{C}) = \mathbb{R} \times Q$$

where a morphism $\gamma : (t_1, q_1) \rightarrow (t_2, q_2)$ is a path

$$\gamma : [t_1, t_2] \rightarrow Q \text{ s.t. } \gamma(t_1) = q_1,$$

Here we've been assuming $t_1 \leq t_2$, so this is not a groupoid. We could adjoin inverses formally to get a groupoid, but then we'd get morphisms like:

\[ 
\begin{array}{c}
\gamma \\
\downarrow \\
[t_1, t_2]
\end{array}
\]

which do indeed show up in Feynman diagrams involving antimatter, but would require further thought.

**Research topics:**

1) Study Feynman's original work on path integrals for a special relativistic particle and see if he allowed paths like this:
If so, formalize what he did using some category \( C \). Is it a groupoid or merely a \(*\)-category?

**Def:** A \(*\)-category is a category \( C \) with a contravariant functor \( * : C \to C \) that's the identity on objects & has \( ** = 1_C \).

Equivalently, for any morphism \( \gamma : x \to y \) there's a morphism \( \gamma^* : y \to x \) s.t.

1. \( (\gamma_1 \circ \gamma_2)^* = \gamma_2^* \circ \gamma_1^* \)
2. \( (\gamma^*)^* = \gamma \)

(these imply \( 1_x^* = 1_x \) \( \forall x \in C \))

This is sometimes called a "category with involution", or in quantum computer science, a "\(*\)-category".

The main example is Hilb, the category of Hilbert spaces & bounded linear operators: given \( T : H \to H' \) we get \( T^* : H' \to H \) defined by

\[
\langle T^* \phi, \psi \rangle = \langle \phi, T \psi \rangle \quad \forall \phi \in H, \psi \in H'
\]
3) $\langle -, - \rangle$ is usually degenerate, but that's OK: we can form

$$K \leq \text{Fun}(\text{Ob}(C))$$

by

$$K = \{ \psi : \langle \phi, \psi \rangle = 0 \quad \forall \phi \in \text{Fun}(\text{Ob}(C)) \}$$

and form the quotient space

$$H_0 = \text{Fun}(\text{Ob}(C))/K$$

on which we have $\langle -, - \rangle$ defined by

$$\langle [\phi], [\psi] \rangle := \langle \phi, \psi \rangle$$

and this is nondegenerate.

4) Is $\langle \psi, \psi \rangle \geq 0$? To get $H_{\psi}$, we need some extra conditions... but we'd need to look at some examples to find nice sufficient conditions. This is somehow related to "reflection positivity" in the Osterwalder–Schwinger Theorem. If we get $\langle \psi, \psi \rangle \geq 0$, we can complete $H_0$ & get a Hilbert space.
Besides the issue of Hilbert spaces, there's the issue of operators. How can we get some nice operators on $\text{Fun}(\text{Ob}(C))$? We can get them from elements $F \in \text{Fun}(\text{Mor}(C))$, some space of "nice" functions on $\text{Mor}(C)$:

$$(F \psi)(y) = \int_{\gamma : x \to y} F(\gamma) \psi(x) \, d\gamma \, dx$$

where "nice" means this converges. In fact we get an algebra of such operators, with some luck:

$$(GF)(\gamma) = \int_{\tilde{\gamma}_1, \tilde{\gamma}_2} G(\tilde{\gamma}_2) F(\tilde{\gamma}_1) \, d\tilde{\gamma}_1 \, d\tilde{\gamma}_2$$

st. $\tilde{\gamma}_2 \tilde{\gamma}_1 = \gamma$

This is "convolution"; $\text{Fun}(\text{Mor}(C))$ is called the "category algebra" of $C$. 