The Big Picture

Last time we sketched how to get a Hilbert space from a category $C$ (of "configurations" & "processes") equipped with an "amplitude" functor

$$A : C \rightarrow U(1)$$

There are lots of subtleties involving analysis, but these evaporate when $C$ is finite -- let's consider this case for simplicity. Then we form a vector space $\text{Fun}(\text{Ob}(C))$ -- which now means all functions

$$\psi : \text{Ob}(C) \rightarrow \mathbb{C}.$$  

$\text{Fun}(\text{Ob}(C))$ is isomorphic to $\mathbb{C}[\text{Ob}(C)]$ -- the space of formal linear combinations of objects of $C$. Then we define a sesquilinear map

$$\langle \cdot, \cdot \rangle : \mathbb{C}[\text{Ob}(C)] \times \mathbb{C}[\text{Ob}(C)] \rightarrow \mathbb{C}$$

linear in this argument

conjugate linear in this argument

by

$$\langle y, x \rangle = \sum_{y : x \rightarrow y} A(y) \quad x, y \in \text{Ob}(C)$$

(and extending to $\mathbb{C}[\text{Ob}(C)]$ by linearity.)
We're doing a "path integral", but now it's a mere sum over morphisms — we're implicitly using counting measure on $\text{Hom}(x,y)$. This inner product may be degenerate. Take $C[\text{Ob}(C)]$ & mod out by

$$\{ \psi \in C[\text{Ob}(C)] : \langle \psi, \phi \rangle = 0 \ \forall \phi \in C[\text{Ob}(C)] \}$$

to get a vector space $H$ with nondegenerate sesquilinear form on it; if that's positive definite, then $H$ is a Hilbert space.

Alex Hoffnung and J.B. have been considering examples like this: "a particle on a (discretized) line":

```
. . . . . .
. . . . .
. . . (t+1,y)
. . (t,x) . .
. . . . .
```

$\text{Ob}(C) = \{1, \ldots, T\} \times \{1, \ldots, X\}$

Morphisms in $C$ are freely generated by morphisms

$$\gamma : (t,x) \rightarrow (t+1,y)$$

$\forall t \in \{1, \ldots, T-1\} \ \forall x, y \in \{1, \ldots, X\}$
So a typical morphism in $\mathbb{C}$ looks like

![Diagram](image)

(A category freely generated in this way is called a "quiver"). If you choose the amplitude $A: C \rightarrow U(1)$ to be a discretized version of the amplitude for a particle on a line, we recover standard physics in the continuum limit.

We really want to categorify all this...
PARTICLES

A category $C$

\[ \begin{array}{c}
\text{\textbullet} \rightarrow \text{\textbullet} \\
y \sim \text{pull of a} \\
\text{particle} \end{array} \]

A functor

$A : C \rightarrow U(1) \otimes C$

the category with one object $*$

elts of $U(1)$ as morphisms

STRINGS

A 2-category or a double category

\[ \begin{array}{c}
\Sigma \nearrow \downarrow \searrow \\
x \sim \text{worldsheet of} \\
\text{a string} \end{array} \]

or

\[ \begin{array}{c}
\Sigma \nearrow \downarrow \searrow \\
\delta_2 \sim \text{boundary} \\
\delta_1 \end{array} \]

A 2-functor

$A : C \rightarrow U(1)[1]$

"shifted $U(1)$"

the 2-category with one object $*$

one morphism $1_*$

elts of $U(1)$ as 2-morphisms

For any abelian group $A$ & any $n \geq 0$, we can form an $n$-category $A[n]$ - the "$n$-times shifted" version of $A$

$U(1)[1] \cong U(1) \text{Tor}$

the monoidal category of $U(1)$-torsors.

For any group $G$ a $G$-set that's isomorphic to $G$, where $G$ acts on itself by left multiplication. "A torus is a group that's forgotten its identity".
If $G$ is abelian, $G\text{-}\text{Tor}$ is a monoidal category with

$$X \otimes Y = \frac{X \times Y}{(xg, y) \sim (x, yg)}$$

where $X, Y \in G\text{-}\text{Tor}$ & $g \in G$ acts on the right on $X$ since $G$ acts on the left and $G$ is abelian.

$U(1)\text{-}\text{Tor}$ has

- one object, $*$
- $U(1)$-tensors as morphisms, with
  - $\otimes$ as composition
  - $U(1)$-tensor morphisms as 2-morphisms.

$U(1)[1]$ is a skeleton of $U(1)\text{-}\text{Tor}$, so in particular $U(1)[1] \simeq U(1)\text{-}\text{Tor}$.

From $A : C \to U(1)$ we try to build a Hilbert space, but first we form the vector space $\text{Fun}(\text{Ob}(C))$, which if $C$ is finite is just $\text{Hom}(\text{Ob}(C), C) \cong C[\text{Ob}(C)]$.

From $A : C \to U(1)\text{-}\text{Tor}$ we try to build a 2-Hilbert space, but first we form the 2-vector space $\text{FUN}(\text{Ob}(C))$, which if $C$ is finite is just $\text{Hom}(\text{Ob}(C), \text{Vect}_c) \cong \text{Vect}_c[\text{Ob}(C)]$. 
We define \( \langle -, - \rangle \) on \( \mathcal{C}[\mathcal{O}(\mathcal{C})] \) by:

\[
\langle y, x \rangle = \sum_{y : x \to y} A(y) \in \mathcal{C}
\]

Here we use

\[
\mathcal{U}(1) \hookrightarrow \mathcal{C}
\]

to add elts of \( \mathcal{U}(1) \) and get elts of \( \mathcal{C} \).

\( \langle -, - \rangle \) on \( \mathbf{Vect}[\mathcal{O}(\mathcal{C})] \) should satisfy

\[
\langle y, x \rangle = \bigoplus_{y : x \to y} A(y) \in \mathbf{Vect}_\mathcal{C}
\]

Here we use

\[
\mathcal{U}(1) \mathbf{Tor}_\mathcal{C} \to \mathbf{Vect}_\mathcal{C}
\]

sending \( \mathcal{U}(1) \) torsors to their corresponding 1-dimensional vector spaces

\[
0 \to \mathcal{O}
\]

--- in fact this is a Hilbert space.

For more, see Daniel Freed's "Higher algebraic structures & quantization."