2-Categories from Typed \(\lambda\)-calculi, cont.

Last time we listed 6 rewrite rules which Lambek & Scott used to create a poset of terms of any given type in any typed \(\lambda\)-calculus, \(P\). They prove (something stronger than this version of) the

\underline{Church-Rosser Theorem}: Given any set of types, there's the initial typed \(\lambda\)-calculus \(P_0\) with those types (\& nothing else other than what the definition requires). The corresponding 2-category \(\hat{C}_{P_0}\) with:

- types as objects
- \((x \in X, \varphi(x))\) (with \(\varphi(x)\) a term of type \(A\)) as morphisms \(f : X \to A\)
- rewrite rules \(\varphi \Rightarrow \psi\) generating 2-morphisms.
- all possible equations between 2-morphisms

has the property that for any objects (types) \(X \& Y\), the category \(\text{hom}(X, Y)\) is a poset, and \underline{this poset is terminating \& confluent}. 

Note: Lambek & Scott's 2-category \( \tilde{C}p \) is just a 2-poset, i.e. a 2-category s.t. the hom categories are all posets. We could also create a more interesting 2-category \( \tilde{C}p \) by letting rewrite rules freely generate the hom categories \( \text{Hom}(X,Y) \). J.B. believes the Church-Rosser Thm. would still hold, i.e. \( \text{Hom}(X,Y) \) would still be terminating and confluent.

Last time we drew surface diagrams for all but two of Lambek & Scott's rewrite rules:

5) \( \beta \)-reduction \( (x \in X \mapsto \varphi(x))(c) \Rightarrow \varphi(c) \quad \varphi(x) \in A \quad c \in X \)

6) \( \eta \)-reduction \( (x \in X \mapsto f(x)) \Rightarrow f \quad f \in \text{hom}(X,A) \)

Let's consider (5) in the example where \( P = \lambda \text{Th(Calc)} \). For example:

\( (x \in R \mapsto x+1)(y) \Rightarrow y+1 \)
Terms give us morphisms, e.g.

\[(xeR, x+1)\]

gives a morphism from \( R \to R \) in \( \mathcal{C}_{\text{Add}}(Gle) \).

As a string diagram, this morphism looks like:

\[
\begin{array}{c}
\text{R} \\
\downarrow \\
\text{R} \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{1} \\
\downarrow \\
\text{R} \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{1} : 1 \to \text{R} \\
\downarrow \\
\text{R} \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{+} : \text{R} \times \text{R} \to \text{R} \\
\downarrow \\
\text{R} \\
\end{array}
\]

Similarly, \((ze1, xeR \to x+1)\) gives a morphism from \( 1 \) to \( \text{hom}(R, R) \) — this is just the curried version of the previous morphism from \( R \) to \( R \). As a string diagram this looks like:

\[
\begin{array}{c}
\text{R} \\
\downarrow \\
\text{R} \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{1} \\
\downarrow \\
\text{R} \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{+} : \text{R} \times \text{R} \to \text{R} \\
\downarrow \\
\text{R} \\
\end{array}
\]

\[
\begin{array}{c}
\text{R} \\
\downarrow \\
\text{R} \\
\end{array}
\]

\[
\begin{array}{c}
\text{R} \\
\downarrow \\
\text{R} \\
\end{array}
\]

Next, \((y \in R, (x \mapsto x+1)(y))\) gives a morphism from \(R\) to \(R\), which has string diagram:

\[
\text{ev} : \text{hom}(R,R) \times R \to R
\]

Our rewrite rule 5 — \(\beta\)-reduction — simplifies this to \((y \in R, y+1)\) which we've seen the diagram for before:
So, β-reduction (in this example) can be drawn as a surface diagram:

In general, β-reduction looks like this:
We see here Thom's "fold catastrophe":

\[ f_t(x) = x^3 + tx \]

\[ t = -1 \quad \text{2 critical pts.} \]
\[ t = 0 \quad \text{1 critical pt.} \]
\[ t = +1 \quad \text{0 critical pts.} \]

If we draw "time" \( t \) in the vertical direction, we get:

\[ t = -1 \]
\[ t = 0 \quad \text{the catastrophe!} \]
\[ t = +1 \]

Thom has a philosophy in which all verbs (processes) correspond to different catastrophes, the fold being the simplest. So: we've seen that "evaluation of a function" (\( \beta \)-reduction) corresponds to the simplest catastrophe.
What about $\eta$-reduction?

6) $x \in X \mapsto f(x) \Rightarrow f \in \text{hom}(X,A)$

This gives a 2-morphism in $C_p$:

$$(y \in 1, x \in X \mapsto f(x)) \Rightarrow (y \in 1, f)$$

Not quite sure how to draw this, but maybe:

Challenge: draw this as a surface. What catastrophe do you get?