Recall from last quarter that de-groupification takes a groupoid to its zeroth homotopy.

\[ X \mapsto H_0(X) = \text{Free}(\text{components of } X) \]

It also takes spans between groupoids to linear operators.

We'll consider an example. Let \( A \) be any abelian category. So we consider \( \text{Ab} \).

Of course, these are closed under \( \oplus = \bigoplus \), so they are never finite unless \( A = \{ 0 \} \).

Essentially, all small abelian categories are \( \cong \mathbf{R}\text{-Mod} \).

And the name comes from the prototype \( \mathbf{Z}\text{-Mod} = \text{category of abelian groups} \).

But we'll try to not get "too infinite" eg. think of the following example:

\( A = \) fin. dim. reps of a finite quiver over a finite field.

The idea is to try to get linear operators \( V \times V \to V \) (equivalently \( V \otimes V \to V \)).

And this will come from a \( \text{tri-span} \)
More precisely, we will de-groupoidify
and get a "3-index tensor"
which can be thought of/re-interpreted as a bi-linear map $V \otimes V \rightarrow V$.

In fact, $?? = \text{SES}(A_0)$.

This is a category, and we have:

$$\begin{array}{ccc}
0 & \rightarrow & S \\
\text{(sub)} & \rightarrow & M \\
\text{(mod)} & \rightarrow & Q \rightarrow 0 \\
\text{(quot)} & \rightarrow & \end{array}$$

Our trio-span now becomes

Now, \[
\begin{bmatrix}
  \alpha & 0 \\
  \beta & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  \xi \\
  (a + b)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \xi \\
  (a + b)
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  \xi \\
  (a + b)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \xi \\
  (a + b)
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  \xi \\
  (a + b)
\end{bmatrix}
\]
and by our $H_0$ - Construction this gives a linear map

$$H_0(D \times 2) \rightarrow H_0(G)$$

and

$$\cong H_0(D) \otimes H_0(G).$$

(e) So **UPSHOT**, we seek from $Ses \Rightarrow A_0$, a linear map

$$H_0(A_0) \otimes H_0(M_0) \rightarrow H_0(M_0)$$

AND hope that this map $m$ is "associative" which would make $H_0(A_0)$ into an assoc. alg.

In fact this does turn out to be the case, and $H_0(A_0)$ is called the **Hall algebra of the abelian category $A$**.

(f) Here's a "rough" reason why we expect the mul. to be associative:

"Multi" these things is like a module of length 3

is a module with a 3-stage filtration

$$0 \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3.$$

But now assoc is

$$\left( a \begin{array}c b \\ c \end{array} \right) = a \left( b \begin{array}c c \end{array} \right)$$

$$M_0 \otimes M_1 \subseteq M_2 \otimes M_3$$

$$M_0 \otimes (M_1 \otimes M_2) = (M_0 \otimes M_1) \otimes M_2.$$
Note that \( H_0(A_0) = \text{decomposition of } \mathbb{R} \), so the fact that \( \sqrt{A} \) has over mult is a "taming down" of the fact that S.E.S. 3-step filtrations are "nice."

2) So our aim now is to try and figure out what exactly \( H_0(A_0) \) is. We start with a very easy example, and then the other example where \( A \) is NOT semisimple.

\( A_0 = \text{Fin Dim Vector} (\text{Any field } F) \implies \cdots \implies H_0(A_0) = F[x] \)

Then it turns out that \( (H_0(A_0), m) = \text{usual polynomial algebra} \).

\( A_0 = \text{Fin Dim Reps of the quiver } Q = (\rightarrow) \text{ over } \text{fixed ground field } F. \)

We first note: \( \text{quiver reps } = (V \xrightarrow{H} W) \)

Moreover, to classify these reps, it is enough to classify the indecomposables and here they are:

\( F \xrightarrow{1} F, \quad 0 \rightarrow F, \quad F \rightarrow 0 \) (of obtaining the null alg.)

\( \square \text{NOTE! This content can be done for all quivers with fin many irreducible reps, and by a remarkable result, these are precisely the simply-laced Dynkin quivers (ADE).} \)

And null alg is indep of orientation in this case.