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1) (a) We continue with our discussion of Hall algebras of abelian categories, obtained via de-categorifying a tri-span — alternately, categorifying Hall algebras leads to tri-spans (among other things).

We'll focus on the example we started to look at last time — that of Rep(A_2) ($\bullet \to \bullet$).

(b) In some sense, Hall algebras arise out of treating the "twisted sum" / extension (of objects in an abelian category) as an "operation". But we're trying to make it an operation using the tool called a "span".
(Spans aren't maps, but "twisted" maps, to use the same analogy.)

(c) From last time — we ended by listing ~~the~~ indecomposable reps of (A_2) over a fixed arbitrary field F. These are

$$0 \xrightarrow{0} F, \quad F \xrightarrow{1} F, \quad F \xrightarrow{0} 0$$

Call these $\boxed{A}$, $\boxed{C}$, $\boxed{B}$, say.

All fin-dim A_2-reps are ~~finite~~ direct sums of A, B, C.
eg: $[F \xrightarrow{0} F] = A \oplus B$

(d) Then the abelian category $\boxed{\text{F.D. } \mathrm{Rep}(A_2)}$ is NOT semisimple, eg

$$0 \to A \to C \to B \to 0$$

Remark: Is it ~~so~~ $0 \to B \to C \to A \to 0$?

Note that in the latter case, we need

$$\begin{array}{ccc} F & \xrightarrow{?} & F \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1} \\ 0 & \xrightarrow{0} & F \end{array}$$

But to be equal / commute, $? = 0$.
But then this is NOT injective, unlike the first term of a short exact sequence!

So, it's not $0 \to B \to C \to A \to 0$, ~~but~~ and in fact,

$0 \to A \to C \to B \to 0$ does work.

ⓔ Note: ~~Thus~~, C is not irreducible.
But, A & B are irreducible — in fact, A & B are the only (fin. dim.) A_2-irreps.

② ⓐ $\boxed{\text{Hall algebra Calculations}}$

Note that $\boxed{\begin{aligned} Hall(\mathcal{A}) &:= H_0(\text{underlying groupoid of } \mathcal{A}) \\ &= H_0(\mathcal{A}_0) \end{aligned}}$

ⓑ Now we focus on our specific example of $FD\,Rep(A_2) = \widetilde{\mathcal{A}}$

But every object in $\widetilde{\mathcal{A}}$ is $\cong$ to $n_1 A \oplus n_2 B \oplus n_3 C$
and $(n_1, n_2, n_3) \longleftrightarrow$ 1-1 corresp. with isoclasses.

So, $H_0(\widetilde{\mathcal{A}}) \cong F[X, Y, Z]$ as vector spaces.

$\boxed{\text{Question}}$ What is the mult. in $H_0(\widetilde{\mathcal{A}})$? Clearly, it can't be commutative because it's NOT $0 \to B \to C \to A \to 0$

but it is $0 \to A \to C \to B \to 0$!

So, what is $H_0(\tilde{A})$ as an algebra?

(c) Well, A_2 is also a Dynkin diagram (connected) $\to$ hence it gives a simple (split) Lie algebra sl_3.

It turns out that $H_0(\tilde{A})$ is suspiciously similar to

$$U_q(n^+) \subseteq U_q(sl_3), \text{ where } sl_3 = n^+ \oplus h \oplus n^-.$$

(d) Let's now do an explicit calculation. Let's compute $A \cdot B$

i.e. $(1,0,0) \cdot (0,1,0)$.

Now there's ~~the~~ Ext^1-group $Ext^1(B,A)$: which is an F-vector space. So we should be looking at a linear combination of all possible elements of $Ext^1(B,A)$.

However, these are all isomorphic to either $A \oplus B$ or to C.

(But the maps are different.)

So we have $(1,0,0) \cdot (0,1,0) = ?(0,0,1) + ??(1,0,1)$

But ? would come from the contributions from every nonzero/nonsplit extension.

So to make everything proper, we need to work only over a <u>finite field</u> $\mathbb{F}_q$

(e) <u>Recall</u> our definition of multiplication in the Hall algebra.
(There are standard texts on this topic, of course, that use Ext^1 to give/define the product etc.)

We used the tri-span

$$\begin{tikzcd}
\boxed{A}\ \mathcal{A} & & \mathcal{A}\ \boxed{B} \\
& SES(\mathcal{A}) \arrow[ul, "S"'] \arrow[ur, "Q"] \arrow[d, "M"] & \\
& \mathcal{A}\ \boxed{C} &
\end{tikzcd}$$

We'll continue (and finish!) this computation next time.

(3) Brief, informal remarks on why mult. in $V = H_0(\mathcal{A}_0)$ is <u>associative</u>: we want

$$\begin{tikzcd}
V \otimes V \otimes V \arrow[r, "M \otimes 1"] \arrow[d, "1 \otimes M"'] \arrow[dr, phantom, "\circlearrowleft"] & V \otimes V \arrow[d, "M"] \\
V \otimes V \arrow[r, "M"] & V
\end{tikzcd}$$

to commute. In our setting, M is a <u>tri-span</u>.