Note that $U_q(g)$-modules have dimension $\dim V_q^n$ (e.g., $\dim V_q^n = \dim V_q(A_2)$), though their $q$-dim may depend only on $q$.

This phenomenon of "$q$-universality" is not quite reflected in a Hall-module from last time. Roughly speaking, $B(W)$ given $W \in \text{Fundim Rep}_{F_q} (A_2)$ was the category of all $V_1 \rightarrow V_2 \rightarrow W$.

But now morphisms in $B(W)$ are $V_1 \rightarrow V_2 \rightarrow W$.

So we must have $\text{Im}(\alpha) = \text{Im}(\beta)$.

But even then, we have many subspaces in $W$ depending on $q$. So the classes of isomorphisms depend on $q$.

To amend this, we now construct a new Hall-module.
We start by replacing $B(V)$ by

$$\text{Mon}^0(V) = \left\{ (V \to W), y \mapsto A \right\},$$

where $A$ has $\text{Rep}_{\text{End}}(A^2)$.

We will now construct the underlying groupoid

whose degroupoidification is the "nice" Hall-module.

Its $(\text{Mon}_{\text{const}}(V))_0 = (\text{monomorphisms over constant reps}).$

**Def:** A constant rep. of a quiver is a vector space $X$ (to be thought of, via $v \mapsto v_x$ at every vertex $v \in X$). Every rep. (edge) is id$_X$.

Its objects in $\text{Mon}_{\text{const}}(V)_0$ are $\text{Rep}_{\text{const}}: V_i \to V_j \to W$.

We have the "constant" functor sending $V_i \to W$.

On the other hand, given any Young diagram $D$, one has the notion of $D$-sheaf as a vector space $X$, and the forgetful functor that takes this to $X$. 
We can now construct the weak/homotopy pullback

\[ \text{Man}_\text{Const}(U) \triangleleft \{ \text{Fg-V-Sp. w/ D-flag} \} \]

"Constant"

\[ (\text{Vec SP}_{F_q}) \]

Thus, essentially \( \mathcal{P} = \text{(Vec Spaces equipped with a) pair of structures} \)

\( a \) "quiver-up embeds it"

\( b \) a D-flag on it

So, we're really trying to compute the "real" homotopy fiber product because these two properties/structures are independent.

(Recall how the fiber product works:

\[ T \times_{B} T = \bigcap_{b \in B} \pi^{-1}(b) \times_{\pi(B)} \pi^{-1}(b) \]

But if we look over with groupoids, then we are allowed to share..."
\[ F : G_1 \to G_2 \]
\[ R \to S \quad \text{or} \quad R \to S', \phi \]

So, the fiber should be replaced in this setting by the \textit{homotopy/mold} fiber, i.e.
\[ F^{-1}(S) = \chi(R, S', \phi) : F(R) = S', \phi \]

And weak pullbacks are also called \textit{homotopy pullbacks}.
\[ = \text{homotopy (fiber product)} \]
\[ = \text{homotopy fiber) product} \]

\( \) (a) There are several things to do now:

(i) Give an example
(ii) Counting the isoclasses (are there only finitely many?)
(iii) Defining the groupification of the action map.

\[ \text{SES}(P, A) \]

Let's do (iii) first.
\[ \text{SES}(P, A) \]

Where
\[ \text{SES}(P, A) \]

is just as
Also note that comparing the types of flags is exactly what (i) is being done in the groupoid \( P \)!

(ii) we also did last quarter, while to look at Hecke operators!

We consider an example:

\[
\begin{align*}
A_2 & \xrightarrow{\text{Sh}_3} \text{Vec} \; V = \mathbb{V}_2 \oplus \mathbb{V}_1 \\
\mathbb{V}_3 & \cong \mathbb{V}_2 \oplus \mathbb{V}_1
\end{align*}
\]

So, the space \( V/\text{D-flag} \) is 2-dim \( \in \) 3-dim \( \mathbb{V}_2 \oplus \mathbb{V}_3 \).

Now combine the 2 structures, and check/compare!
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<th>dim B</th>
<th># classes</th>
<th># total</th>
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</table>

**Total = dim (module) = 18**

**Q:** Why is this a module for $U_q(g)$? All we know is that it's a module over the algebra $U_q(g)$.